

Generalized Fredholm theory in semisimple algebras

D. MÄNNLE – C. SCHMOEGER

RIASSUNTO: Sia \mathcal{A} una algebra complessa semisemplice con identità $e \neq 0$. Sia $\Phi_g(\mathcal{A})$ la sottoclasse formata dagli elementi $x \in \mathcal{A}$ che verificano la seguente condizione:

$\exists y \in \mathcal{A}$: tale che $xyx = x$, e inoltre $e - xy - yx$ è un elemento di Fredholm.

Ogni elemento di Fredholm appartiene a $\Phi_g(\mathcal{A})$. Si studia la classe $\Phi_g(\mathcal{A})$ i cui elementi sono detti elementi di Fredholm generalizzati.

ABSTRACT: Let \mathcal{A} be a semisimple complex algebra with identity $e \neq 0$. We write $\Phi_g(\mathcal{A})$ for the following class of elements of \mathcal{A} .

$\Phi_g(\mathcal{A}) = \{x \in \mathcal{A} : \exists y \in \mathcal{A} \text{ such that } xyx = x \text{ and } e - xy - yx \text{ is Fredholm}\}$.

Each Fredholm element of \mathcal{A} belongs to $\Phi_g(\mathcal{A})$. Elements in $\Phi_g(\mathcal{A})$ we call generalized Fredholm elements. In this paper we investigate the class $\Phi_g(\mathcal{A})$.

1 – Introduction

In this paper we always assume that \mathcal{A} is a complex algebra with identity $e \neq 0$. If X is a complex Banach space, then it is well known that $\mathcal{L}(X) = \{T : X \rightarrow X : T \text{ is linear and bounded}\}$ is a semisimple Banach algebra.

In [1] S. R. CARADUS has introduced the class of *generalized Fredholm operators*. $T \in \mathcal{L}(X)$ is called a generalized Fredholm operator, if

there is some $S \in \mathcal{L}(X)$ with $TST = T$ and $I - TS - ST$ is a Fredholm operator. This class of operators is studied in [15], [16] and [17].

If \mathcal{A} is semisimple, *generalized Fredholm elements* in \mathcal{A} are introduced in [10] as follows: $x \in \mathcal{A}$ is called a generalized Fredholm element if there is some $y \in \mathcal{A}$ such that $xyx = x$ and $e - xy - yx$ is a Fredholm element in \mathcal{A} . Some of the results in [15] and [16] are generalized in [10].

The present paper is an improvement and a continuation of [10]. Furthermore we generalize some of the results in [17].

In Section 2 of this paper we collect some results concerning relatively regular elements in algebras. Section 3 contains a summary of Fredholm theory in semisimple algebras. In section 4 we investigate generalized invertible elements. This concept will be useful in the next sections, where we present the main results of this paper.

In Section 5 we study algebraic properties of generalized Fredholm elements. Section 6 contains a characterization of Riesz elements in complex semisimple Banach algebras and a result concerning the stability of generalized Fredholm elements under holomorphic functional calculus. Section 7 contains various results on ascent and descent and a “punctured neighbourhood theorem” for generalized Fredholm elements.

2 – Relatively regular elements

An element $x \in \mathcal{A}$ is called *relatively regular*, if $xyx = x$ for some $y \in \mathcal{A}$. In this case y is called a *pseudo-inverse* of x .

PROPOSITION 2.1. *For $x \in \mathcal{A}$ the following assertions are equivalent:*

- (1) x is relatively regular.
- (2) There is $y \in \mathcal{A}$ with $xyx = x$ and $yxy = y$.
- (3) There is $p = p^2 \in \mathcal{A}$ with $x\mathcal{A} = p\mathcal{A}$.
- (4) There is $q = q^2 \in \mathcal{A}$ with $\mathcal{A}x = \mathcal{A}q$.

PROOF. (1) \Rightarrow (2): Suppose that $xy_0x = x$. Put $y = y_0xy_0$. Then it is easy to see that $xyx = x$ and $yxy = y$.

(2) \Rightarrow (1): Clear.

(1) \Rightarrow (3): Take $y \in \mathcal{A}$ with $xyx = x$ and put $p = xy$. Then $x\mathcal{A} = xyx\mathcal{A} \subseteq p\mathcal{A} = xy\mathcal{A} \subseteq x\mathcal{A}$.

(3) \Rightarrow (1): We have $p = xa$ for some $a \in \mathcal{A}$ and $x = px$, thus $x = px = (xa)x = xax$.

Similar arguments as above show that (1) and (4) are equivalent. \square

PROPOSITION 2.2. *Suppose that $x, u \in \mathcal{A}$, $xux - x$ is relatively regular and that r is a pseudo-inverse of $xux - x$. Then x is relatively regular and*

$$y = u - r + uxr + rxu - uxrux$$

is a pseudo-inverse of x .

PROOF. From $(xux - x)r(xux - x) = xux - x$, we get

$$\begin{aligned} x &= xux - uxrxux + uxrx + xrxux - xrx = \\ &= x(u - uxrux + uxr + rxu - r)x = xyx. \end{aligned} \quad \square$$

For $x \in \mathcal{A}$ we define

$$R(x) = \{a \in \mathcal{A} : xa = 0\} \text{ and } L(x) = \{a \in \mathcal{A} : ax = 0\}.$$

The proof of the next proposition is easy and left to the reader.

PROPOSITION 2.3. *Suppose that $x \in \mathcal{A}$ is relatively regular and y is a pseudo-inverse of x . Then xy , yx , $e - xy$ and $e - yx$ are idempotent and*

$$\begin{aligned} xy\mathcal{A} &= x\mathcal{A}, \quad \mathcal{A}yx = \mathcal{A}x, \\ R(x) &= (e - yx)\mathcal{A}, \quad L(x) = \mathcal{A}(e - xy). \end{aligned}$$

A proof for the following result can be found in [6, p. 15].

PROPOSITION 2.4. *If $x \in \mathcal{A}$ is relatively regular, $xyx = x$ and $yx = y$, then we have for $z \in \mathcal{A}$:*

z is a pseudo-inverse of x if and only if there is some $u \in \mathcal{A}$ with

$$z = y + u - yxuxy.$$

3 – Fredholm theory in semisimple algebras

Throughout this section we assume that \mathcal{A} is semisimple. This means that $\text{rad}(\mathcal{A}) = \{0\}$, where $\text{rad}(\mathcal{A})$ denotes the radical of \mathcal{A} . For the convenience of the reader we shall summarize some concepts of the Fredholm theory in algebras. See [2]-[4], [11]-[14], [18]-[20] for details.

We call an element $e_0 \in \mathcal{A}$ *minimal idempotent*, if $e_0\mathcal{A}e_0$ is a division algebra and $e_0^2 = e_0$. $\text{Min}(\mathcal{A})$ denotes the set of all minimal idempotents of \mathcal{A} .

PROPOSITION 3.1. (1) *Suppose that $\mathcal{R} \subseteq \mathcal{A}[\mathcal{L} \subseteq \mathcal{A}]$ is a right [left] ideal in \mathcal{A} . Then $\mathcal{R}[\mathcal{L}]$ is a minimal right [left] ideal if and only if $\mathcal{R} = e_0\mathcal{A}[\mathcal{L} = \mathcal{A}e_0]$ for some $e_0 \in \text{Min}(\mathcal{A})$.*

(2) *If $\text{Min}(\mathcal{A}) \neq \emptyset$, then the sum of all minimal right ideals equals the sum of all minimal left ideals.*

PROOF. (1) [4, B.A. 3.1], (2) [5, Prop. 30.10.]. □

The *socle* of \mathcal{A} , $\text{soc}(\mathcal{A})$, is defined to be the sum of all minimal right ideals if $\text{Min}(\mathcal{A}) \neq \emptyset$. If $\text{Min}(\mathcal{A}) = \emptyset$, then we set $\text{soc}(\mathcal{A}) = \{0\}$. Proposition 3.1 shows that

$$(3.2) \quad \text{soc}(\mathcal{A}) \text{ is an ideal of } \mathcal{A},$$

and

$$(3.3) \quad \text{Min}(\mathcal{A}) \subseteq \text{soc}(\mathcal{A}).$$

From now on we always assume in this section that $\text{soc}(\mathcal{A}) \neq \{0\}$.

Suppose that $\mathcal{J} \subseteq \mathcal{A}$ is a right [left] ideal of \mathcal{A} . \mathcal{J} has *finite order* if \mathcal{J} can be written as the sum of a finite number of minimal right [left] ideals of \mathcal{A} . The *order* $\Theta(\mathcal{J})$ of \mathcal{J} is defined to be the smallest number of minimal right [left] ideals which have sum \mathcal{J} . We define $\Theta(\{0\}) = 0$ and $\Theta(\mathcal{J}) = \infty$, if \mathcal{J} does not have finite order.

PROPOSITION 3.4. *Suppose that \mathcal{J} and \mathcal{K} are right [left] ideals of \mathcal{A} and $n \in \mathbb{N}$.*

$$(1) \quad \Theta(\mathcal{J}) < \infty \Leftrightarrow \mathcal{J} \subseteq \text{soc}(\mathcal{A}).$$

(2) $\Theta(\mathcal{J}) = n$, if and only if there are $e_1, \dots, e_n \in \text{Min}(\mathcal{A})$ such that $e_i e_j = 0$ for $i \neq j$ and

$$\begin{aligned} \mathcal{J} &= (e_1 + \dots + e_n)\mathcal{A} = e_1\mathcal{A} \oplus \dots \oplus e_n\mathcal{A} \\ [\mathcal{J} &= \mathcal{A}(e_1 + \dots + e_n) = \mathcal{A}e_1 \oplus \dots \oplus \mathcal{A}e_n] . \end{aligned}$$

(3) If $\Theta(\mathcal{K}) < \infty$, $\mathcal{J} \subseteq \mathcal{K}$ and $\mathcal{J} \neq \mathcal{K}$ then $\Theta(\mathcal{K}) < \Theta(\mathcal{J})$.

(4) $\Theta(x\mathcal{A}) = \Theta(\mathcal{A}x)$ for each $x \in \mathcal{A}$.

(5) $\text{soc}(\mathcal{A}) = \{x \in \mathcal{A} : \Theta(x\mathcal{A}) < \infty\}$.

PROOF. (1) Clear. (2) and (3): [2, §2]. (4) and (5): [9]. \square

DEFINITIONS.

(1) For $x \in \mathcal{A}$ we define the *nullity* of x by

$$\text{nul}(x) = \Theta(R(x))$$

and the *defect* of x by

$$\text{def}(x) = \Theta(L(x)) .$$

(2) The group of the invertible elements of \mathcal{A} is denoted by \mathcal{A}^{-1} .

(3) The quotient algebra $\mathcal{A}/\text{soc}(\mathcal{A})$ is denoted by $\widehat{\mathcal{A}}$. For $x \in \mathcal{A}$ we write $\widehat{x} = x + \text{soc}(\mathcal{A})$ for the coset of x in $\widehat{\mathcal{A}}$.

(4) The set of *Fredholm elements* of \mathcal{A} is given by

$$\Phi(\mathcal{A}) = \{x \in \mathcal{A} : \widehat{x} \in \widehat{\mathcal{A}}^{-1}\} .$$

The next proposition contains some useful characterisations of Fredholm elements.

PROPOSITION 3.5. For $x \in \mathcal{A}$ the following assertions are equivalent:

(1) $x \in \Phi(\mathcal{A})$.

(2) There are $p, q \in \text{soc}(\mathcal{A})$ such that $p = p^2$, $q = q^2$ and

$$\mathcal{A}x = \mathcal{A}(e - p), \quad x\mathcal{A} = (e - q)\mathcal{A} .$$

(3) x is relatively regular and $R(x), L(x) \subseteq \text{soc}(\mathcal{A})$.

(4) x is relatively regular and $\text{nul}(x), \text{def}(x) < \infty$.

(5) x is relatively regular and for each pseudo-inverse y of x we have $\widehat{xy} = \widehat{e} = \widehat{yx}$.

(6) x is relatively regular and there is a pseudo-inverse y of x such that $\widehat{xy} = \widehat{e} = \widehat{yx}$.

PROOF. (1) \Leftrightarrow (2) [4, F.1.10].

(2) \Rightarrow (3): It is easy to see that $R(x) = p\mathcal{A}$ and $L(x) = q\mathcal{A}$. Thus $R(x), L(x) \subseteq \text{soc}(\mathcal{A})$.

(3) \Leftrightarrow (4) \Leftrightarrow (5): Suppose that y is a pseudo-inverse of x . Proposition 2.3 gives

$$R(x) = (e - yx)\mathcal{A} \text{ and } L(x) = \mathcal{A}(e - xy) .$$

Therefore we get from Proposition 3.4 (1):

$$\begin{aligned} R(x), L(x) \subseteq \text{soc}(\mathcal{A}) &\Leftrightarrow \Theta(R(x)), \Theta(L(x)) < \infty \Leftrightarrow \\ &\Leftrightarrow \text{nul}(x), \text{def}(x) < \infty \Leftrightarrow e - yx, e - xy \in \text{soc}(\mathcal{A}) . \end{aligned}$$

(5) \Rightarrow (6): Clear.

(6) \Rightarrow (1): From $\widehat{xy} = \widehat{e} = \widehat{yx}$ we get $\widehat{x} \in \widehat{\mathcal{A}}^{-1}$, thus $x \in \Phi(\mathcal{A})$. \square

The *index* of $x \in \Phi(\mathcal{A})$ is defined by

$$\text{ind}(x) = \text{nul}(x) - \text{def}(x) .$$

A proof of the next result can be found in [19, Theorem 4.5 and Theorem 4.6].

THEOREM 3.6. *If $x, y \in \Phi(\mathcal{A})$ and $s \in \text{soc}(\mathcal{A})$ then*

- (1) $xy \in \Phi(\mathcal{A})$ and $\text{ind}(xy) = \text{ind}(x) + \text{ind}(y)$;
- (2) $x + s \in \Phi(\mathcal{A})$ and $\text{ind}(x + s) = \text{ind}(x)$;
- (3) *If \mathcal{A} is a Banach algebra then there are $\delta > 0$ and $\alpha, \beta \in \mathbb{N}_0$ such that*
 - (i) $x + u \in \Phi(\mathcal{A})$, $\text{ind}(x + u) = \text{ind}(x)$, $\text{nul}(x + u) \leq \text{nul}(x)$ and $\text{def}(x + u) \leq \text{def}(x)$ for all $u \in \mathcal{A}$ with $\|u\| < \delta$.
 - (ii) $\text{nul}(\lambda e - x) = \alpha \leq \text{nul}(x)$ and $\text{def}(\lambda e - x) = \beta \leq \text{def}(x)$ for $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \delta$.

The ideal of *inessential elements* of \mathcal{A} is given by

$$I(\mathcal{A}) = \bigcap \{P : P \text{ is a primitive ideal of } \mathcal{A} \text{ with } \text{soc}(\mathcal{A}) \subseteq P\}.$$

We write $\tilde{\mathcal{A}}$ for the quotient algebra $\mathcal{A}/I(\mathcal{A})$ and \tilde{x} for the coset $x + I(\mathcal{A})$ of $x \in \mathcal{A}$.

PROPOSITION 3.7.

- (1) $\text{soc}(\mathcal{A}) \subseteq I(\mathcal{A})$.
- (2) $x \in \Phi(\mathcal{A}) \Leftrightarrow \tilde{x} \in \tilde{\mathcal{A}}^{-1}$.
- (3) If \mathcal{A} is a Banach algebra, then $I(\mathcal{A})$ is closed.

PROOF. (1) Clear. (2) [4, F.3.2]. (3) Each primitive ideal of a Banach algebra is closed. \square

PROPOSITION 3.8. *Let $s \in \text{soc}(\mathcal{A})$. Then s is relatively regular and there is $b \in \text{soc}(\mathcal{A})$ such that*

$$sbs = s \quad \text{and} \quad bsb = b.$$

PROOF. From Proposition 3.4 we get $e_1, \dots, e_n \in \text{Min}(\mathcal{A})$ with $e_i e_j = \delta_{ij} e_i$ and

$$s\mathcal{A} = (e_1 + \dots + e_n)\mathcal{A} = e_1\mathcal{A} \oplus \dots \oplus e_n\mathcal{A}.$$

Put $p = e_1 + \dots + e_n$. Then $s\mathcal{A} = p\mathcal{A}$ and $p^2 = p$. Proposition 2.1 shows that s is relatively regular, hence there is $a \in \mathcal{A}$ with $sas = s$. Put $b = asa$. Then $sbs = s$ and $bsb = b$. \square

Now we are ready to introduce the class of generalized Fredholm elements. First we give some examples.

EXAMPLES 3.9. (1) Let $s \in \text{soc}(\mathcal{A})$. By Proposition 3.8 there is $b \in \text{soc}(\mathcal{A})$ such that $sbs = s$. Hence

$$(e - sb - bs) + \text{soc}(\mathcal{A}) = \hat{e} \in \hat{\mathcal{A}}^{-1}$$

thus

$$e - sb - bs \in \Phi(\mathcal{A}).$$

(2) Let $x \in \Phi(\mathcal{A})$. Proposition 3.5 gives $\widehat{x}\widehat{y} = \widehat{e} = \widehat{y}\widehat{x}$ for each pseudo-inverse y of x . Thus

$$(e - xy - yx) + \text{soc}(\mathcal{A}) = -\widehat{e} \in \widehat{\mathcal{A}}^{-1}$$

hence

$$e - xy - yx \in \Phi(\mathcal{A}) .$$

(3) If $x \in \mathcal{A}^{-1}$ and $y = x^{-1}$, then $xyx = x$ and

$$e - xy - yx = -e \in \mathcal{A}^{-1} \subseteq \Phi(\mathcal{A}) .$$

(4) Let $x \in \mathcal{A}$ with $\mathcal{A} = x\mathcal{A} \oplus R(x)$ or $\mathcal{A} = \mathcal{A}x \oplus L(x)$. Theorem 3.3 in [15] shows that there exists $y \in \mathcal{A}$ such that $xyx = x$ and $xy = yx$. Therefore $e - xy - yx = e - 2xy$ and $(e - 2xy)^2 = e - 4xy + 4xyxy = e$. Thus

$$e - xy - yx \in \mathcal{A}^{-1} \subseteq \Phi(\mathcal{A}) .$$

(5) Let $x \in \mathcal{A}$ with $x^2 = x$. Put $y = x$. Then $xyx = x$ and $e - xy - yx = e - 2x$. From $(e - 2x)^2 = e$ we get

$$e - xy - yx \in \mathcal{A}^{-1} \subseteq \Phi(\mathcal{A}) .$$

In each of the above examples the elements $x \in \mathcal{A}$ has the following property: there is a pseudo-inverse y of x such that $e - xy - yx \in \Phi(\mathcal{A})$.

Therefore we call an element $x \in \mathcal{A}$ a *generalized Fredholm element* if x is relatively regular and there is a pseudo-inverse y of x with $e - xy - yx \in \Phi(\mathcal{A})$. By $\Phi_g(\mathcal{A})$ we denote the set of all generalized Fredholm elements of \mathcal{A} .

Before we state our first results concerning the class $\Phi_g(\mathcal{A})$ we need the following lemma.

LEMMA 3.10. *Suppose that $x, u \in \mathcal{A}$.*

- (1) *If $xux - x \in \mathcal{A}^{-1}$ then $x \in \mathcal{A}^{-1}$.*
- (2) *If $xux - x \in \Phi(\mathcal{A})$ then $x \in \Phi(\mathcal{A})$.*

PROOF. (1) Put $v = (xux - x)^{-1}$, $x_1 = v(xu - e)$ and $x_2 = (ux - e)v$. Then $x_1x = v(xu - e)x = v(xux - x) = e$ and $xx_2 = x(ux - e)v = (xux - x)v = e$.

(2) Since $\widehat{x}\widehat{u}\widehat{x} - \widehat{x} \in \widehat{\mathcal{A}}^{-1}$, it follows from (1) that $\widehat{x} \in \widehat{\mathcal{A}}^{-1}$, thus $x \in \Phi(\mathcal{A})$. \square

THEOREM 3.11.

(1) $\text{soc}(\mathcal{A}) \subseteq \Phi_g(\mathcal{A})$.

(2) $\Phi(\mathcal{A}) \subseteq \Phi_g(\mathcal{A})$.

(3) If $x \in \Phi(\mathcal{A})$ and if y is a pseudo-inverse of x , then $e - xy - yx \in \Phi(\mathcal{A})$ and $\text{ind}(e - xy - yx) = 0$.

(4) If \mathcal{A} is a Banach algebra and $x \in \Phi_g(\mathcal{A})$, then there is $\delta > 0$ such that

$$\lambda e - x \in \Phi_g(\mathcal{A}) \text{ for all } \lambda \in \mathbb{C} \text{ with } |\lambda| < \delta$$

and

$$\lambda e - x \in \Phi(\mathcal{A}) \text{ for all } \lambda \in \mathbb{C} \text{ with } 0 < |\lambda| < \delta .$$

PROOF. (1) follows from Example 3.9 (1).

(2) follows from Example 3.9 (2).

(3) From Example 3.9 (2) we get $s \in \text{soc}(\mathcal{A})$ such that $e - xy - yx = -e + s$. Use Theorem 3.6 (2) to derive

$$\text{ind}(e - xy - yx) = \text{ind}(-e + s) = \text{ind}(-e) = 0 .$$

(4) Take $y \in \mathcal{A}$ such that $xyx = x$ and $v = e - xy - yx \in \Phi(\mathcal{A})$. For $\lambda \in \mathbb{C}$ put

$$w(\lambda) = (\lambda e - x)(y + \lambda^2 e)(\lambda e - x) + (\lambda e - x) .$$

An easy computation gives

$$w(\lambda) = \lambda(\lambda(\lambda e - x)^2 + \lambda y + v) .$$

Since $\Phi(\mathcal{A})$ is open (Theorem 3.6 (3)), there is $\gamma > 0$ such that $v + u \in \Phi(\mathcal{A})$ for all $u \in \mathcal{A}$ with $\|u\| < \gamma$. There is $\delta > 0$ such that

$$\|\lambda(\lambda e - x)^2 + \lambda y\| < \gamma \quad \text{for all } \lambda \in \mathbb{C} \text{ with } |\lambda| < \delta .$$

Thus $\frac{1}{\lambda}w(\lambda) \in \Phi(\mathcal{A})$ for $0 < |\lambda| < \delta$. This gives $w(\lambda) \in \Phi(\mathcal{A})$ for $0 < |\lambda| < \delta$. From Lemma 3.10 we get $\lambda e - x \in \Phi(\mathcal{A})$ if $0 < |\lambda| < \delta$.

THEOREM 3.12. *For $x \in \mathcal{A}$ the following assertions are equivalent:*

- (1) $x \in \Phi_g(\mathcal{A})$.
- (2) There is $y \in \mathcal{A}$ such that $\widehat{x}\widehat{y}\widehat{x} = \widehat{x}$ and $\widehat{e} - \widehat{x}\widehat{y} - \widehat{y}\widehat{x} \in \widehat{\mathcal{A}}^{-1}$.

PROOF. (1) \Rightarrow (2) Clear.

(2) \Leftarrow (1): Since $xyx - x \in \text{soc}(\mathcal{A})$, $xyx - x$ is relatively regular (Proposition 3.8). Proposition 2.2 shows that x is relatively regular and that

$$y_0 = y - r + yxr + rxy - yxrxy$$

is a pseudo-inverse of x , where r is a pseudo-inverse of $xyx - x$. Then we get

$$\begin{aligned} \widehat{x}\widehat{y}_0 &= \widehat{x}\widehat{y} - \widehat{x}\widehat{r} + \underbrace{\widehat{x}\widehat{y}\widehat{x}\widehat{r}}_{=\widehat{x}} + \widehat{x}\widehat{r}\widehat{x}\widehat{y} - \underbrace{\widehat{x}\widehat{y}\widehat{x}\widehat{r}\widehat{x}\widehat{y}}_{=\widehat{x}} \\ &= \widehat{x}\widehat{y}. \end{aligned}$$

A similar argument shows that $\widehat{y}_0\widehat{x} = \widehat{y}\widehat{x}$. We summarize: $xy_0x = x$ and

$$\widehat{e} - \widehat{x}\widehat{y}_0 - \widehat{y}_0\widehat{x} = \widehat{e} - \widehat{x}\widehat{y} - \widehat{y}\widehat{x} \in \widehat{\mathcal{A}}^{-1}.$$

Thus $x \in \Phi_g(\mathcal{A})$. □

Let \mathcal{B} be a complex algebra with identity $e \neq 0$. In view of Theorem 3.12 it seems to be useful to consider elements $t \in \mathcal{B}$ with the following property:

- (3.13) t is relatively regular and for some pseudo-inverse s of t the element $e - ts - st$ belongs to \mathcal{B}^{-1} .

Therefore we define

$$\mathcal{B}^g = \{t \in \mathcal{B} : t \text{ has the property (3.13)}\}.$$

Elements in \mathcal{B}^g can be called *generalized invertible*, since $\mathcal{B}^{-1} \subseteq \mathcal{B}^g$. Observe that $0 \in \mathcal{B}^g$, thus \mathcal{B}^{-1} is a proper subset of \mathcal{B}^g . With these notations we have

$$x \in \Phi(\mathcal{A}) \Leftrightarrow \widehat{x} \in \widehat{\mathcal{A}}^{-1}$$

and, by Theorem 3.12,

$$(3.14) \quad x \in \Phi_g(\mathcal{A}) \Leftrightarrow \hat{x} \in \hat{\mathcal{A}}^g.$$

In the next section we shall investigate the class \mathcal{B}^g .

4 – Properties of \mathcal{B}^g

In this section \mathcal{B} always denotes a complex algebra with identity e .

Let V be a vector space and $T : V \rightarrow V$ linear. For the definitions of the *ascent* $p(T)$ and the *descent* $q(T)$ of T we refer the reader to [8, §72].

Let $t \in \mathcal{B}$ and let the linear operators $L_t, R_t : \mathcal{B} \rightarrow \mathcal{B}$ be defined by

$$L_t(b) = tb \text{ and } R_t(b) = bt \text{ (} b \in \mathcal{B} \text{)}.$$

Then we have $L_t(\mathcal{B}) = t\mathcal{B}$, $R_t(\mathcal{B}) = \mathcal{B}t$, $\text{kern}L_t = R(t)$ and $\text{kern}R_t = L(t)$.

Put

$$\begin{aligned} p_l(t) &= p(L_t), & q_l(t) &= q(L_t) \\ p_r(t) &= p(R_t), & q_r(t) &= q(R_t). \end{aligned}$$

PROPOSITION 4.1. *Let $t \in \mathcal{B}$.*

(1) *If $p_l(t)$ and $q_l(t)$ [$p_r(t)$ and $q_r(t)$] are both finite, then they are equal and for $n = p_l(t)$ [$n = p_r(t)$] we have*

$$\mathcal{B} = R(t^n) \oplus t^n \mathcal{B} \quad [\mathcal{B} = L(t^n) \oplus \mathcal{B}t^n].$$

(2) $p_r(t) \leq q_l(t)$, $p_l(t) \leq q_r(t)$.

PROOF. (1) [8, §72].

(2) We only show that $p_r(t) \leq q_l(t)$. If $n = q_l(t) < \infty$ then $t^n \mathcal{B} = t^{n+1} \mathcal{B}$. Take $b \in \mathcal{B}$ with $t^n = t^{n+1}b$. If $c \in L(t^{n+1})$ then $ct^n = ct^{n+1}b = 0$, thus $c \in L(t^n)$, therefore $L(t^{n+1}) \subseteq L(t^n)$, thus $p_r(t) \leq n$. \square

From [15, Theorem 3.3] we get the following characterization of elements of \mathcal{B}^g .

PROPOSITION 4.2. *For $t \in \mathcal{B}$ the following assertions are equivalent.*

- (1) $t \in \mathcal{B}^g$.
- (2) $p_l(t) = q_l(t) \leq 1$.
- (3) $p_r(t) = q_r(t) \leq 1$.
- (4) *There is $u \in \mathcal{B}$ with $tut = t$ and $tu = ut$.*

COROLLARY 4.3. *Let \mathcal{A} be a complex semisimple algebra with identity. Suppose that $0 < p = p_l(x) = q_l(x) < \infty$. Then $x^p \in \Phi_g(\mathcal{A})$.*

PROOF. Since $p_l(x^p) = q_l(x^p) \leq 1$, it follows from Proposition 4.2, that $x^p u x^p = x^p$ and $x^p u = u x^p$ for some $u \in \mathcal{A}$. Then we get $e - x^p u - u x^p = e - 2u x^p$ and $(e - 2u x^p)^2 = e$. Thus

$$e - x^p u - u x^p \in \mathcal{A}^{-1} \subseteq \Phi(\mathcal{A}) . \quad \square$$

PROPOSITION 4.4. *Suppose that $t, u, t_1, t_2 \in \mathcal{B}$.*

- (1) *If $t_1, t_2 \in \mathcal{B}^g$, $t_1 t_2 = t_1 t_2$ then $t_1 t_2 \in \mathcal{B}^g$.*
- (2) *If $t \in \mathcal{B}^g$ and $n \in \mathbb{N}$ then $t^n \in \mathcal{B}^g$.*
- (3) *If $t \in \mathcal{B}^g$ then there is a unique $s \in \mathcal{B}$ with*

$$tst = t, \quad sts = s \text{ and } ts = st .$$

Furthermore we have $s \in \mathcal{B}^g$ and if $ta = at$ for some $a \in \mathcal{B}$, then $sa = as$.

- (4) *If $t, u \in \mathcal{B}^g$ and $tu = ut = 0$, then $t + u \in \mathcal{B}^g$.*
- (5) $e - t_1 t_2 \in \mathcal{B}^g \iff e - t_2 t_1 \in \mathcal{B}^g$.

PROOF. (1) follows from Proposition 3.4 in [15] and 4.2.

(2) follows from (1).

(3) Proposition 3.9 in [15] shows that there is a unique $s \in \mathcal{B}$ such that $tst = t$, $sts = s$ and $ts = st$. From Proposition 4.2 (4) we get $s \in \mathcal{B}^g$. If $ta = at$, then

$$sta = sat = satst = sat^2s = st^2as = tas ,$$

thus $s^2ta = tas^2$ and therefore

$$sa = stsa = s^2ta = tas^2 = ats^2 = as .$$

(4) From (3) we get $s, v \in \mathcal{B}$ such that $tst = t$, $sts = s$, $ts = st$, $uvu = u$, $vuv = v$ and $uv = vu$. Then

$$\begin{aligned} (t+u)(s+v) &= ts + tv + us + uv = ts + tvuv + usts + uv = \\ &= ts + tuv^2 + uts^2 + uv = ts + uv . \end{aligned}$$

A similar computation gives $(s+v)(t+u) = st + vu$. Thus $(t+u)(s+v) = (s+v)(t+u)$. From Proposition 4.2 we get $t+u \in \mathcal{B}^g$ since

$$\begin{aligned} (t+u)(s+v)(t+u) &= (ts+uv)(t+u) = tst + tsu + uvt + uvu = \\ &= t + stu + vut + u = t + u . \end{aligned}$$

(5) We only have to show that $e - t_1t_2 \in \mathcal{B}^g$ implies $e - t_2t_1 \in \mathcal{B}^g$. By Proposition 4.2 (4) there is a pseudo-inverse s of $e - t_1t_2$ which commutes with $e - t_1t_2$. Put $r = e + t_2st_1$. A simple computation shows that r is a pseudo-inverse of $e - t_2t_1$ which commutes with $e - t_2t_1$. \square

NOTATIONS. Let \mathcal{B} be a Banach algebra and $t \in \mathcal{B}$. By $\sigma(t)$ and $r(t)$ we denote the spectrum and the spectral radius of t , respectively. If $D \subseteq \mathbb{C}$ is open, $\sigma(t) \subseteq D$ and $f : D \rightarrow \mathbb{C}$ holomorphic, then $f(t)$ is defined by the well-known operational calculus.

PROPOSITION 4.5. *Suppose that \mathcal{B} is a Banach algebra and $t \in \mathcal{B}^g$.*

(1) *t is quasinilpotent if and only if $t = 0$.*

(2) If $t \neq 0$ and if s is the unique pseudo-inverse of t with the properties in part (3) of Proposition 4.4, then

- (i) s is not quasinilpotent;
- (ii) $\sigma(s) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : \frac{1}{\lambda} \in \sigma(t)\}$;
- (iii) $t \in \mathcal{B}^{-1}$ or 0 is a pole of order 1 of $(\lambda e - t)^{-1}$;
- (iv) $\text{dist}(0, \sigma(t) \setminus \{0\}) = r(s)^{-1}$.

(3) Suppose that $0 \in \sigma(t)$, $D \subseteq \mathbb{C}$ is a region, $\sigma(t) \subseteq D$, $f : D \rightarrow \mathbb{C}$ holomorphic, injective and $f(0) = 0$. Then $f(t) \in \mathcal{B}^g$.

PROOF. [16, Propositions 2.6, 2.7 and 2.13]. □

PROPOSITION 4.6. Let $\lambda_1, \dots, \lambda_m$ pairwise distinct complex numbers. If $t \in \mathcal{B}$ and

$$\prod_{j=1}^m (t - \lambda_j e) = 0,$$

then $t - \lambda e \in \mathcal{B}^g$ for each $\lambda \in \mathbb{C}$.

PROOF. [16, Proposition 2.4]. □

5 – Algebraic properties of $\Phi_g(\mathcal{A})$

As in Section 3 we denote by \mathcal{A} a complex semisimple algebra with identity. Furthermore we assume $\{0\} \neq \text{soc}(\mathcal{A})$.

THEOREM 5.1. $\Phi_g(\mathcal{A}) + \text{soc}(\mathcal{A}) \subseteq \Phi_g(\mathcal{A})$.

PROOF. Let $x \in \Phi_g(\mathcal{A})$ and $s \in \text{soc}(\mathcal{A})$. Then $\widehat{x+s} = \widehat{x} + \widehat{s} = \widehat{x}$. By (3.14), $\widehat{x} \in \widehat{\mathcal{A}}^g$, thus $\widehat{x+s} \in \widehat{\mathcal{A}}^g$, hence $x+s \in \Phi_g(\mathcal{A})$. □

REMARK. From Proposition 3 (2) we know that

$$\Phi(\mathcal{A}) + I(\mathcal{A}) \subseteq \Phi(\mathcal{A}).$$

In Section 6 of this paper we shall see that in general

$$\Phi_g(\mathcal{A}) + I(\mathcal{A}) \not\subseteq \Phi_g(\mathcal{A}).$$

THEOREM 5.2. (1) *If $x_1, x_2 \in \Phi_g(\mathcal{A})$ and $x_1x_2 - x_2x_1 \in \text{soc}(\mathcal{A})$, then $x_1x_2 \in \Phi_g(\mathcal{A})$.*

(2) *If $x \in \Phi_g(\mathcal{A})$ and $n \in \mathbb{N}$, then $x^n \in \Phi_g(\mathcal{A})$.*

PROOF. (1) By (3.14), $\widehat{x}_1, \widehat{x}_2 \in \widehat{\mathcal{A}}^g$. Since $\widehat{x}_1\widehat{x}_2 = \widehat{x}_2\widehat{x}_1$, we get from Proposition 4.4 (1), that $\widehat{x}_1\widehat{x}_2 = \widehat{x}_1\widehat{x}_2 \in \widehat{\mathcal{A}}^g$. By (3.14), $x_1x_2 \in \Phi_g(\mathcal{A})$.

(2) follows from (1). \square

REMARKS. (1) In [15, 1.7 (d)] it is shown by an example, that if $x_1, x_2 \in \Phi_g(\mathcal{A})$ it does not follow that $x_1x_2 \in \Phi_g(\mathcal{A})$.

(2) In [15, 1.7 (b)] it is shown by an example, that if $x^n \in \Phi_g(\mathcal{A})$ for some $n \in \mathbb{N}$ it does not follow that $x \in \Phi_g(\mathcal{A})$.

THEOREM 5.3. *For $x_1, x_2 \in \mathcal{A}$ we have:*

$$e - x_1x_2 \in \Phi_g(\mathcal{A}) \iff e - x_2x_1 \in \Phi_g(\mathcal{A}).$$

PROOF. Proposition 4.4 (5) and (3.14) give

$$\begin{aligned} e - x_1x_2 \in \Phi_g(\mathcal{A}) &\iff \widehat{e} - \widehat{x}_1\widehat{x}_2 \in \widehat{\mathcal{A}}^g \iff \\ &\iff e - x_2x_1 \in \Phi_g(\mathcal{A}). \end{aligned} \quad \square$$

THEOREM 5.4. *For $x \in \mathcal{A}$ the following assertions are equivalent:*

- (1) $x \in \Phi_g(\mathcal{A})$;
- (2) there is $y \in \mathcal{A}$ such that $xyx = x$ and $\widehat{x}\widehat{y} = \widehat{y}\widehat{x}$.

PROOF. (1) \Rightarrow (2): From $\widehat{x} \in \widehat{\mathcal{A}}^g$ it follows that there exists some $u \in \mathcal{A}$ with $\widehat{x}\widehat{u}\widehat{x} = \widehat{x}$ and $\widehat{x}\widehat{u} = \widehat{u}\widehat{x}$ (Proposition 4.2 (4)). Then we have $xux - x \in \text{soc}(\mathcal{A})$. Proposition 3.8 shows that $xux - x$ is relatively regular. Let r be a pseudo-inverse of $xux - x$. Then

$$y = u - r + uxr + rxu - uxrux$$

is a pseudo-inverse of x (Proposition 2.2). Then it is easy to see that $\widehat{x}\widehat{y} = \widehat{x}\widehat{u} = \widehat{u}\widehat{x} = \widehat{y}\widehat{x}$.

(2) \Rightarrow (1) Proposition 4.2 (4) gives $\widehat{x} \in \widehat{\mathcal{A}}^g$, thus $x \in \Phi_g(\mathcal{A})$. \square

THEOREM 5.5. *For $x \in \Phi_g(\mathcal{A})$ the following assertions are equivalent:*

- (1) $x \in \Phi(\mathcal{A})$;
- (2) $\text{nul}(x) < \infty$;
- (3) $\text{def}(x) < \infty$.

PROOF. It is clear that (1) implies (2) and (3).

(2) \Rightarrow (1): Take a pseudo-inverse y of x with $\widehat{y}x = \widehat{x}y$ (Theorem 5.4). Since

$$\text{nul}(x) = \Theta(R(x)) = \Theta((e - yx)\mathcal{A}) < \infty ,$$

we get from Proposition 3.4 (5), that $e - yx \in \text{soc}(\mathcal{A})$, hence $\widehat{e} = \widehat{y}x = \widehat{x}y$, thus $\widehat{x} \in \widehat{\mathcal{A}}^{-1}$.

A similar proof shows that (3) implies (1). □

THEOREM 5.6. *Suppose that $x, u \in \Phi_g(\mathcal{A})$ and that $xu, ux \in \text{soc}(\mathcal{A})$. Then $x + u \in \Phi_g(\mathcal{A})$.*

PROOF. Since $\widehat{x}, \widehat{u} \in \widehat{\mathcal{A}}^g$ and $\widehat{x}\widehat{u} = \widehat{0} = \widehat{u}\widehat{x}$, Proposition 4.4 (4) gives $\widehat{x + u} = \widehat{x} + \widehat{u} \in \widehat{\mathcal{A}}^g$, thus $x + u \in \Phi_g(\mathcal{A})$. □

6 – Topological properties of $\Phi_g(\mathcal{A})$

In this section we assume that \mathcal{A} is complex semisimple Banach algebra with identity $e \neq 0$. From [4, R. 3.6] it follows that

$$\mathcal{A} \neq I(\mathcal{A}) \iff \dim \mathcal{A} = \infty .$$

Hence, if $\dim \mathcal{A} = \infty$, $\widetilde{\mathcal{A}} = \mathcal{A}/I(\mathcal{A})$ is a complex Banach algebra with identity $\widetilde{e} \neq \widetilde{0}$.

The first result in this section is an improvement of Theorem 3.11 (4).

THEOREM 6.1. *Suppose that $x \in \Phi_g(\mathcal{A})$, $z \in \Phi(\mathcal{A})$ and $xz - zx \in \text{soc}(\mathcal{A})$. Then there is $\delta > 0$ such that*

$$x - \lambda z \in \Phi(\mathcal{A}) \text{ for } 0 < |\lambda| < \delta .$$

PROOF. Take $u \in \mathcal{A}$ such that $\widehat{z}\widehat{u} = \widehat{u}\widehat{z} = \widehat{e}$. From $\widehat{x}\widehat{z} = \widehat{z}\widehat{x}$ we get $\widehat{x} = \widehat{x}(\widehat{u}\widehat{z}) = \widehat{x}\widehat{z}\widehat{u} = \widehat{z}\widehat{x}\widehat{u}$, thus $\widehat{u}\widehat{x} = \widehat{u}\widehat{z}\widehat{x}\widehat{u} = \widehat{x}\widehat{u}$, hence $xu - ux \in \text{soc}(\mathcal{A})$. Since $u \in \Phi(\mathcal{A}) \subseteq \Phi_g(\mathcal{A})$, we derive from Theorem 5.2 (1) that $ux \in \Phi_g(\mathcal{A})$. Theorem 3.11 (4) shows that there is $\delta > 0$ such that $ux - \lambda e \in \Phi(\mathcal{A})$ for $0 < |\lambda| < \delta$. This implies, since $z \in \Phi(\mathcal{A})$, that $zux - \lambda z = z(ux - \lambda e) \in \Phi(\mathcal{A})$ ($0 < |\lambda| < \delta$). Then we have for $0 < |\lambda| < \delta$ that

$$\widehat{x} - \lambda\widehat{z} = \widehat{z}\widehat{u}\widehat{x} - \lambda\widehat{z} \in \widehat{\mathcal{A}}^{-1},$$

thus $x - \lambda z \in \Phi(\mathcal{A})$. \square

DEFINITION. Let $x \in \mathcal{A}$. The set

$$\sigma_{\Phi}(x) = \{\lambda \in \mathbb{C} : \lambda e - x \notin \Phi(\mathcal{A})\}$$

is called the *Fredholm spectrum* of x . If $\sigma_{\Phi}(x) = \{0\}$, then x is called a *Riesz element* of \mathcal{A} .

If $\dim \mathcal{A} = \infty$, then by Proposition 3.7 (2)

$$\sigma_{\Phi}(x) = \sigma(\widetilde{x});$$

and (see Theorem 6.1)

$$\text{dist}(0, \sigma_{\Phi}(x) \setminus \{0\}) > 0 \text{ if } x \in \Phi_g(\mathcal{A}).$$

Suppose that $x \in \Phi_g(\mathcal{A})$, then there is $y \in \mathcal{A}$ such that

$$(6.2) \quad xyx = x, \quad yxy = y, \quad \text{and} \quad xy - yx \in \text{soc}(\mathcal{A}).$$

In fact, by Theorem 5.4, $xy_0x = x$ and $xy_0 - y_0x \in \text{soc}(\mathcal{A})$ for some $y_0 \in \mathcal{A}$. Then $y = y_0xy_0$ satisfies (6.2).

THEOREM 6.3. *Suppose that $\dim \mathcal{A} = \infty$, $x \in \Phi_g(\mathcal{A})$ and y is a pseudo-inverse of x which satisfies (6.2).*

(1) *x is a Riesz element $\iff y$ is a Riesz element. In this case $\sigma_{\Phi}(x) = \{0\}$ and $\text{dist}(0, \sigma_{\Phi}(x) \setminus \{0\}) = \infty$.*

(2) If x is not a Riesz element, then

$$\sigma_{\Phi}(y) \setminus \{0\} = \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \frac{1}{\lambda} \in \sigma_{\Phi}(x) \right\}$$

and

$$\text{dist}(0, \sigma_{\Phi}(x) \setminus \{0\}) = r(\tilde{y})^{-1} .$$

PROOF. (1) (6.2) and Proposition 4.2 (4) show that $\tilde{x}, \tilde{y} \in \tilde{\mathcal{A}}^g$. Therefore, by Proposition 4.5 (1) and (6.2)

$$\begin{aligned} x \text{ is Riesz} &\iff r(\tilde{x}) = 0 \iff \tilde{x} = \tilde{0} \iff \tilde{y} = \tilde{0} \iff \\ &\iff r(\tilde{y}) = 0 \iff y \text{ is Riesz} . \end{aligned}$$

(2) From (1) we see that $r(\tilde{x}), r(\tilde{y}) > 0$. Proposition 4.5 (2) (ii) shows that

$$\begin{aligned} \sigma_{\Phi}(y) \setminus \{0\} &= \sigma(\tilde{y}) \setminus \{0\} = \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \frac{1}{\lambda} \in \sigma(\tilde{x}) \right\} = \\ &= \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \frac{1}{\lambda} \in \sigma_{\Phi}(x) \right\} . \end{aligned}$$

From Proposition 4.5 (2) (iv) we conclude that

$$\text{dist}(0, \sigma_{\Phi}(x) \setminus \{0\}) = \text{dist}(0, \sigma(\tilde{x}) \setminus \{0\}) = r(\tilde{y})^{-1} . \quad \square$$

THEOREM 6.4. *Suppose that $\dim \mathcal{A} = \infty$, $x \in \Phi_g(\mathcal{A})$, $0 \in \sigma(x)$, $D \subseteq \mathbb{C}$ is a region, $\sigma(x) \subseteq D$, $f : D \rightarrow \mathbb{C}$ is holomorphic and injective and $f(0) = 0$. Then $f(x) \in \Phi_g(\mathcal{A})$.*

PROOF. It is clear that $\sigma(\tilde{x}) \subseteq \sigma(x)$.

CASE 1. $x \in \Phi(\mathcal{A})$. Hence $0 \notin \sigma(\tilde{x})$. Since f is injective and $f(0) = 0$, we get

$$0 \notin f(\sigma(\tilde{x})) = \sigma(f(\tilde{x})) = \sigma(\widetilde{f(x)}) = \sigma_{\Phi}(f(x)) ,$$

thus $f(x) \in \Phi(\mathcal{A}) \subseteq \Phi_g(\mathcal{A})$.

CASE 2. $x \notin \Phi(\mathcal{A})$. Then $0 \in \sigma(\tilde{x})$. Let y be a pseudo-inverse of x such that (6.2) holds. Since f' is injective, there is a holomorphic function

$g : f(D) \rightarrow \mathbb{C}$ such that $g(f(\lambda)) = \lambda$. Without loss of generality we can assume that $f'(0) = g'(0) = -1$, thus there is $h : f(D) \rightarrow \mathbb{C}$ holomorphic such that

$$g(\lambda) = -\lambda + \lambda^2 h(\lambda) .$$

Put $\varphi = h \circ f$. From

$$\lambda = g(f(\lambda)) = -f(\lambda) + f(\lambda)^2 h(f(\lambda)) = f(\lambda)\varphi(\lambda)f(\lambda) - f(\lambda)$$

we derive

$$(6.5) \quad x = f(x)\varphi(x)f(x) - f(x) .$$

Since x is relatively regular, (6.5) and Proposition 2.2 show that $f(x)$ is relatively regular and that

$$r = \varphi(x) - [\varphi(x)f(x) - e]y[\varphi(x)f(x) - e]$$

is a pseudo-inverse of $f(x)$ (observe that $f(x)\varphi(x) = \varphi(x)f(x)$). From $\tilde{x}\tilde{y} = \tilde{y}\tilde{x}$ we get $\tilde{y}\widetilde{f(x)} = \widetilde{f(x)}\tilde{y}$ and $\tilde{y}\widetilde{\varphi(x)} = \widetilde{\varphi(x)}\tilde{y}$, thus $\tilde{r}\widetilde{f(x)} = \widetilde{f(x)}\tilde{r}$. Hence

$$\begin{aligned} (\tilde{e} - \tilde{r}\widetilde{f(x)} - \widetilde{f(x)}\tilde{r})^2 &= (\tilde{e} - 2\tilde{r}\widetilde{f(x)})^2 = \\ &= \tilde{e} - 4\tilde{r}\widetilde{f(x)} + 4\tilde{r}\underbrace{\widetilde{f(x)}\tilde{r}\widetilde{f(x)}}_{=\widetilde{f(x)}} = \tilde{e} . \end{aligned}$$

This shows that $e - rf(x) - f(x)r \in \Phi(\mathcal{A})$. □

For our next results we denote by $\overline{\mathcal{M}}$ the closure of a subset $\mathcal{M} \subseteq \mathcal{A}$.

THEOREM 6.6. *Let $x \in \mathcal{A}$.*

- (1) *If x is relatively regular then $x\mathcal{A}x$ is closed.*
- (2) *$x \in \text{soc}(\mathcal{A}) \Leftrightarrow \dim x\mathcal{A}x < \infty$.*
- (3) *If x is relatively regular then*

$$x \in \overline{\text{soc}(\mathcal{A})} \Leftrightarrow x \in \text{soc}(\mathcal{A}) .$$

PROOF. (1) Let y be a pseudo-inverse of x . Then $x\mathcal{A} = xy\mathcal{A}$, $\mathcal{A}x = \mathcal{A}yx$, xy and yx are idempotent. It follows that $x\mathcal{A}$ and $\mathcal{A}x$ are closed, thus $x\mathcal{A} \cap \mathcal{A}x$ is closed. To complete the proof we show that

$$x\mathcal{A}x = x\mathcal{A} \cap \mathcal{A}x .$$

The inclusion “ \subseteq ” is clear. Take $z \in x\mathcal{A} \cap \mathcal{A}x$, thus $z \in xy\mathcal{A}$ and $z \in \mathcal{A}yx$, hence $z = xyz$ and $z = zyx$. This gives $z = xy(zyx) = x(yzy)x \in x\mathcal{A}x$.

(2) is shown in [1].

(3) We only have to show the implication “ \Rightarrow ”. There is a sequence (x_n) in $\text{soc}(\mathcal{A})$ such that $\|x_n - x\| \rightarrow 0$ ($n \rightarrow \infty$). Thus there is $\gamma \geq 0$ with $\|x_n\| \leq \gamma$ for all $n \in \mathbb{N}$. Define the bounded linear operators $F_n, K : \mathcal{A} \rightarrow \mathcal{A}$ by

$$F_n a = x_n a x_n \quad \text{and} \quad K a = x a x \quad (a \in \mathcal{A}, n \in \mathbb{N}) .$$

For each $a \in \mathcal{A}$ we get

$$\begin{aligned} \|K a - F_n a\| &= \|(x - x_n) a x + x_n a (x - x_n)\| \leq \\ &\leq \|a\| (\|x\| + \gamma) \|x - x_n\| , \end{aligned}$$

hence $\|K - F_n\| \leq (\|x\| + \gamma) \|x - x_n\|$.

This shows that (F_n) converges uniformly to K . From (2) we get that each F_n is a finite-dimensional operator, hence K is compact. Since x is relatively regular, K has closed range, by (1), hence K has a finite-dimensional range. Now use (2) to get $x \in \text{soc}(\mathcal{A})$. \square

REMARK. From Proposition 3.7 (2) we know that

$$\Phi(\mathcal{A}) + I(\mathcal{A}) \subseteq \Phi(\mathcal{A}) .$$

Now take $x \in \overline{\text{soc}(\mathcal{A})}$. Since $0 \in \Phi_g(\mathcal{A})$, we have $x \in \Phi_g(\mathcal{A}) + \overline{\text{soc}(\mathcal{A})}$. Theorem 6.6 (3) shows that

$$x \in \Phi_g(\mathcal{A}) \Leftrightarrow x \in \text{soc}(\mathcal{A}) .$$

Thus, in general,

$$\Phi_g(\mathcal{A}) + \overline{\text{soc}(\mathcal{A})} \not\subseteq \Phi_g(\mathcal{A}) ,$$

hence

$$\Phi_g(\mathcal{A}) + I(\mathcal{A}) \not\subseteq \Phi_g(\mathcal{A}) .$$

THEOREM 6.7. *Let $\dim \mathcal{A} = \infty$ and let $x \in \Phi_g(\mathcal{A})$. The following assertions are equivalent:*

- (1) x is a Riesz element of \mathcal{A} .
- (2) $x \in I(\mathcal{A})$.
- (3) $x \in \text{soc}(\mathcal{A})$.

PROOF. (1) \Rightarrow (2): If x is a Riesz element then \tilde{x} is quasinilpotent. As in the proof of Theorem 6.3 we have $\tilde{x} \in \tilde{\mathcal{A}}^g$. Proposition 4.5 (1) shows then that $\tilde{x} = \tilde{0}$, hence $x \in I(\mathcal{A})$.

(2) \Rightarrow (3): By $\check{\mathcal{A}}$ we denote the quotient algebra $\mathcal{A}/\overline{\text{soc}(\mathcal{A})}$. Since $\dim \mathcal{A} = \infty$ ($\Leftrightarrow \mathcal{A} \neq I(\mathcal{A})$), $\check{\mathcal{A}}$ is a Banach algebra with identity $\check{e} \neq \check{0}$, where $\check{z} = z + \overline{\text{soc}(\mathcal{A})}$ denotes the coset of z in $\check{\mathcal{A}}$. Take $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\lambda\check{e} - \check{x} = \lambda\check{e} \in \check{\mathcal{A}}^{-1}$. From Proposition 3.7 (2) we get $\lambda e - x \in \Phi(\mathcal{A})$, thus $\lambda\check{e} - \check{x} \in \check{\mathcal{A}}^{-1}$, hence, for some $z \in \mathcal{A}$,

$$z(\lambda e - x) - e, (\lambda e - x)z - e \in \text{soc}(\mathcal{A}) ,$$

therefore $\check{z}(\lambda\check{e} - \check{x}) = \check{e} = (\lambda\check{e} - \check{x})\check{z}$. This gives $\lambda\check{e} - \check{x} \in \check{\mathcal{A}}^{-1}$. Thus, since $\lambda \in \mathbb{C} \setminus \{0\}$ was arbitrary, \check{x} is quasinilpotent. Let y be a pseudo-inverse of x such that (6.2) holds. Proposition 4.2 (4) shows that $\check{x} \in \check{\mathcal{A}}^g$. Now use Proposition 4.5 (1) to derive $\check{x} = \check{0}$. Therefore $x \in \overline{\text{soc}(\mathcal{A})}$. Theorem 6.6 (3) gives $x \in \text{soc}(\mathcal{A})$.

(3) \Rightarrow (1). For each $\lambda \in \mathbb{C} \setminus \{0\}$ we have $\lambda e - x \in \Phi(\mathcal{A}) + \text{soc}(\mathcal{A}) \subseteq \Phi(\mathcal{A})$. □

Let \mathcal{J} be an ideal in \mathcal{A} . \mathcal{J} is called a Φ -ideal, if $\text{soc}(\mathcal{A}) \subseteq \mathcal{J} \subseteq I(\mathcal{A})$. It is clear that $\text{soc}(\mathcal{A})$, $\overline{\text{soc}(\mathcal{A})}$ and $I(\mathcal{A})$ are Φ -ideals and that

$$\Phi(\mathcal{A}) + \mathcal{J} \subseteq \Phi(\mathcal{A})$$

for each Φ -ideal \mathcal{J} .

COROLLARY 6.8. *Suppose that $\dim \mathcal{A} = \infty$, \mathcal{J} is a Φ -ideal and that*

$$\Phi_g(\mathcal{A}) + \mathcal{J} \subseteq \Phi_g(\mathcal{A}) .$$

Then $\mathcal{J} = \text{soc}(\mathcal{A})$.

PROOF. Take $a \in \mathcal{J}$. Then $a = 0 + a \in \Phi_g(\mathcal{A}) + \mathcal{J} \subseteq \Phi_g(\mathcal{A})$. Since $a \in I(\mathcal{A})$, $\lambda\tilde{e} - \tilde{a} = \lambda\tilde{e}$, thus we have that $\lambda e - a \in \Phi(\mathcal{A})$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Therefore a is a Riesz element and $a \in \Phi_g(\mathcal{A})$. Theorem 6.7 gives $a \in \text{soc}(\mathcal{A})$. \square

THEOREM 6.9. $\Phi_g(\mathcal{A}) \subseteq \overline{\Phi(\mathcal{A})}$.

PROOF. Use Theorem 3.11 (4). \square

7 – Ascent and descent of elements in $\Phi_g(\mathcal{A})$

In this section we assume that \mathcal{A} is a complex semisimple Banach algebra with identity e and that $\text{soc}(\mathcal{A}) \neq \{0\}$.

For $x \in \mathcal{A}$ we define

$$\Delta_l(x) = \{\alpha \in \mathbb{N}_0 : R(x) \cap x^\alpha \mathcal{A} = R(x) \cap x^{\alpha+k} \mathcal{A} \text{ for all } k \geq 0\}$$

and

$$\Delta_r(x) = \{\beta \in \mathbb{N}_0 : L(x) \cap \mathcal{A}x^\beta = L(x) \cap \mathcal{A}x^{\beta+k} \text{ for all } k \geq 0\}.$$

PROPOSITION 7.1. *If $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$, then*

- (1) $p_l(x) \leq n \Leftrightarrow R(x) \cap x^n \mathcal{A} = \{0\}$;
- (2) $q_l(x) \leq n \Leftrightarrow R(x^n) + x\mathcal{A} = \mathcal{A}$;
- (3) $q_r(x) \leq n \Leftrightarrow L(x^n) + \mathcal{A}x = \mathcal{A}$;
- (4) $p_r(x) \leq n \Leftrightarrow L(x) \cap \mathcal{A}x^n = \{0\}$;
- (5) $\Delta_l(x) = \{\alpha \in \mathbb{N}_0 : R(x^\alpha) + x\mathcal{A} = R(x^{\alpha+k}) + x\mathcal{A} \text{ for all } k \geq 0\}$;
- (6) $\Delta_r(x) = \{\beta \in \mathbb{N}_0 : L(x^\beta) + \mathcal{A}x = L(x^{\beta+k}) + \mathcal{A}x \text{ for all } k \geq 0\}$.

PROOF. We only show (1), (2) and (5). The proofs for (3), (4) and (6) are similar.

(1) follows from [8, Satz 72.1].

(2) “ \Rightarrow ” By [8, Satz 72.2], there is a subspace \mathcal{U} of \mathcal{A} such that $\mathcal{A} = \mathcal{U} \oplus x\mathcal{A}$ and $\mathcal{U} \subseteq R(x^n)$. Thus $R(x^n) + x\mathcal{A} = \mathcal{A}$.

“ \Leftarrow ”: Take $y \in x^n\mathcal{A}$. Then $y = x^na$ for some $a \in \mathcal{A}$. There are u, v with $a = u + v$, $u \in R(x^n)$ and $v \in x\mathcal{A}$. It follows that $y = x^n(u + v) = x^nv \in x^{n+1}\mathcal{A}$. Hence $q_l(x) \leq n$.

(5) Denote by M the set on the right side in (5). Let $\alpha \in \Delta_l(x)$ and take $z \in R(x^{\alpha+1}) + x\mathcal{A}$, hence $z = u + xv$ with $u \in R(x^{\alpha+1})$ and $v \in \mathcal{A}$. Then $x^\alpha u \in R(x) \cap x^\alpha\mathcal{A} = R(x) \cap x^{\alpha+1}\mathcal{A}$, thus $x^\alpha u = x^{\alpha+1}w$ for some $w \in \mathcal{A}$, hence $u - xw \in R(x^\alpha)$. It follows that $z = u + xv = (u - xw) + x(w + v) \in R(x^\alpha) + x\mathcal{A}$. We have shown that $R(x^{\alpha+1}) + x\mathcal{A} = R(x^\alpha) + x\mathcal{A}$. By induction we see that $\alpha \in M$.

Now let $\alpha \in M$ and take $z \in R(x) \cap x^\alpha\mathcal{A}$. Then there is $y \in \mathcal{A}$ with $z = x^\alpha y$ and $x^{\alpha+1}y = 0$. Thus $y \in R(x^{\alpha+1}) \subseteq R(x^{\alpha+1}) + x\mathcal{A} = R(x^\alpha) + x\mathcal{A}$. Therefore $y = y_1 + y_2$ with $y_1 \in R(x^\alpha)$, $y_2 \in x\mathcal{A}$. Then $z = x^\alpha(y_1 + y_2) = x^\alpha y_2 \in x^{\alpha+1}\mathcal{A}$, thus $z \in R(x) \cap x^{\alpha+1}\mathcal{A}$. We have shown that $R(x) \cap x^\alpha\mathcal{A} = R(x) \cap x^{\alpha+1}\mathcal{A}$. By induction we see that $\alpha \in \Delta_l(x)$. \square

PROPOSITION 7.2. *Let $x \in \Phi_g(\mathcal{A})$. Then*

- (1) $\Theta(R(x) \cap x\mathcal{A}) < \infty$;
- (2) $\Theta(L(x) \cap \mathcal{A}x) < \infty$;
- (3) $\Delta_l(x) \neq \emptyset$ and $\Delta_r(x) \neq \emptyset$.

PROOF. We only show that $\Theta(R(x) \cap x\mathcal{A}) < \infty$ and $\Delta_l(x) \neq \emptyset$. Take a pseudo-inverse y of x such that $v = e - xy - yx \in \Phi(\mathcal{A})$. If $z \in R(x) \cap x\mathcal{A}$, then $z = (e - yx)z = xyz$, hence $vz = z - xyz - yxz = 0$, thus $z \in R(v)$. Therefore $R(x) \cap x\mathcal{A} \subseteq R(v)$. It follows from Proposition 3.4 (3) and 3.5 (4) that $\Theta(R(x) \cap x\mathcal{A}) \leq \Theta(R(v)) = \text{nul}(v) < \infty$. For $n \in \mathbb{N}$ put $\Theta_n = \Theta(R(x) \cap x^n\mathcal{A})$. Since $R(x) \cap x^{n+1}\mathcal{A} \subseteq R(x) \cap x^n\mathcal{A}$ we derive from Proposition 3.4 (3) that

$$0 \leq \cdots \leq \Theta_{n+1} \leq \Theta_n \leq \cdots \leq \Theta_1 < \infty .$$

Since $\Theta_n \in \mathbb{N}_0$ for $n \in \mathbb{N}$, there is some $\alpha \in \mathbb{N}$ such that $\Theta_{\alpha+k} = \Theta_\alpha$ for all $k \geq 0$. Use Proposition 3.4 (3) to see that $R(x) \cap x^\alpha \mathcal{A} = R(x) \cap x^{\alpha+k} \mathcal{A}$ for all $k \geq 0$. Hence $\alpha \in \Delta_l(x)$. \square

In view of Proposition 7.2 (3) we define for $x \in \Phi_g(\mathcal{A})$:

$$\delta_l(x) = \min \Delta_l(x) \quad \text{and} \quad \delta_r(x) = \min \Delta_r(x).$$

PROPOSITION 7.3. *For $x \in \Phi_g(\mathcal{A})$ we have*

(1) $p_l(x) = q_r(x)$ and $q_l(x) = p_r(x)$.

(2) *If $\alpha = \delta_l(x)$ then*

$$p_l(x) < \infty \Leftrightarrow R(x) \cap x^\alpha \mathcal{A} = \{0\}.$$

In this case $p_l(x) = \delta_l(x)$.

(3) *If $\beta = \delta_r(x)$ then*

$$q_l(x) < \infty \Leftrightarrow L(x^\beta) + \mathcal{A}x = \mathcal{A}.$$

In this case $q_l(x) = \delta_r(x)$.

PROOF. (1) Proposition 4.1 (2) gives $p_l(x) \leq q_r(x)$. Without loss of generality we assume that $n = p_l(x) < \infty$. Since $x^n, x^{n+1} \in \Phi_g(\mathcal{A})$, x^n and x^{n+1} are relatively regular. Thus $\mathcal{A}x^n = \mathcal{A}p$, $\mathcal{A}x^{n+1} = \mathcal{A}q$ for some $p = p^2$, $q = q^2 \in \mathcal{A}$. Then it follows that

$$(e - p)\mathcal{A} = R(x^n) = R(x^{n+1}) = (e - q)\mathcal{A},$$

thus $e - q = (e - p)(e - q) = e - q - p + pq$, hence $p = pq$. Then $\mathcal{A}x^n = \mathcal{A}p = \mathcal{A}pq \subseteq \mathcal{A}q = \mathcal{A}x^{n+1} \subseteq \mathcal{A}x^n$. Hence $q_r(x) \leq n = p_l(x)$.

The proof for $q_l(x) = p_r(x)$ is similar.

(2) “ \Rightarrow ”: Put $p = p_l(x)$. Proposition 7.1 (1) gives $R(x) \cap x^p \mathcal{A} = \{0\}$, thus $p \in \Delta_l(x)$ and $\alpha \leq p$.

“ \Leftarrow ” follows from Proposition 7.1 (1).

(3) Similar. \square

For the rest of this section we always assume that \mathcal{A} is a Banach algebra.

For our further investigation the following concepts will be useful.

For $x \in \mathcal{A}$ we define

$$a_l(x) = \{\mu \in \mathbb{C}: \text{there is a neighbourhood } U \text{ of } \mu \text{ and a holomorphic function } f : U \rightarrow \mathcal{A} \text{ such that } (\lambda e - x)f(\lambda) = 0 \text{ on } U \text{ and } f(\mu) \neq 0\}.$$

$$a_r(x) = \{\mu \in \mathbb{C}: \text{there is a neighbourhood } U \text{ of } \mu \text{ and a holomorphic function } f : U \rightarrow \mathcal{A} \text{ such that } f(\lambda)(\lambda e - x) = 0 \text{ on } U \text{ and } f(\mu) \neq 0\}.$$

It is clear that $a_l(x)$ and $a_r(x)$ are open subsets of \mathbb{C} .

PROPOSITION 7.4. *Let $x \in \mathcal{A}$.*

- (1) *If $\lambda_0 \in a_l(x)$ then $p_l(\lambda_0 e - x) = q_l(\lambda_0 e - x) = \infty$.*
- (2) *If $\lambda_0 \in a_r(x)$ then $p_r(\lambda_0 e - x) = q_r(\lambda_0 e - x) = \infty$.*

PROOF. [13, Theorem 3.5]. □

LEMMA 7.5. *Let $u \in \Phi(\mathcal{A})$.*

- (1) $p_l(u) = 0 \Leftrightarrow \text{nul}(u) = 0$.
- (2) $q_l(u) = 0 \Leftrightarrow \text{def}(u) = 0$.

PROOF. (1) $p_l(u) = 0 \Leftrightarrow R(u) = \{0\} \Leftrightarrow \text{nul}(u) = 0$.

(2) Use Proposition 7.3 (1) to get

$$q_l(u) = 0 \Leftrightarrow p_r(u) = 0 \Leftrightarrow L(u) = \{0\} \Leftrightarrow \text{def}(u) = 0. \quad \square$$

In Theorem 3.11 (4) we have seen that if $x \in \Phi_g(\mathcal{A})$, then there is $\delta > 0$ such that $\lambda e - x \in \Phi(\mathcal{A})$ for $0 < |\lambda| < \delta$. This and Theorem 3.6 (3) (ii) show

PROPOSITION 7.6. *If $x \in \Phi_g(\mathcal{A})$, then there is $\epsilon > 0$ and there are $n, m \in \mathbb{N}_0$ such that*

$$\text{nul}(\lambda e - x) = n \text{ and } \text{def}(\lambda e - x) = m \quad \text{for } 0 < |\lambda| < \epsilon.$$

THEOREM 7.7. *Let $x \in \Phi_g(\mathcal{A})$.*

(1) *If $p_l(x) (= q_r(x)) < \infty$, then there is $\epsilon > 0$ such that*

$$p_l(\lambda e - x) = \text{nul}(\lambda e - x) = 0 \text{ for } 0 < |\lambda| < \epsilon .$$

(2) *If $q_l(x) (= p_r(x)) < \infty$, then there is $\epsilon > 0$ such that*

$$q_l(\lambda e - x) = \text{def}(\lambda e - x) = 0 \text{ for } 0 < |\lambda| < \epsilon .$$

(3) *The following assertions are equivalent:*

(i) $p_l(x) (= q_r(x)) = \infty$.

(ii) $0 \in a_l(x)$.

(iii) *There is $\epsilon > 0$ with $\text{nul}(\lambda e - x) > 0$ for $|\lambda| < \epsilon$.*

(4) *The following assertions are equivalent:*

(i) $q_l(x) (= p_r(x)) = \infty$.

(ii) $0 \in a_r(x)$.

(iii) *There is $\epsilon > 0$ with $\text{def}(\lambda e - x) > 0$ for $|\lambda| < \epsilon$.*

PROOF. We only prove (1) and (3).

(1) Define the bounded linear operator $T : \mathcal{A} \rightarrow \mathcal{A}$ by $Ta = xa$ ($a \in \mathcal{A}$). Then $p(T) = p_l(x)$ and $T^n(\mathcal{A}) = x^n\mathcal{A}$ ($n \in \mathbb{N}$). Since $x^n \in \Phi_g(\mathcal{A})$, x^n is relatively regular, thus $T^n(\mathcal{A})$ is closed. Lemma 2.5 in [9] shows that there is $\epsilon > 0$ with $p(\lambda I - T) = 0$ for $0 < |\lambda| < \epsilon$. Use Lemma 7.5 to conclude that (1) holds.

(3) (i) \Rightarrow (ii): Write $\mathcal{M} = \bigcap_{n=1}^{\infty} x^n\mathcal{A}$. As above, each $x^n\mathcal{A}$ is closed, thus \mathcal{M} is a closed subspace of \mathcal{A} . We have

$$(7.8) \quad x\mathcal{M} = \mathcal{M} .$$

In fact, since the inclusion “ \subseteq ” is clear, we only have to show that $\mathcal{M} \subseteq x\mathcal{M}$. Since $\Delta_l(x) \neq \emptyset$ (Proposition 7.2), there is $\alpha \in \mathbb{N}_0$ with

$$R(x) \cap x^\alpha\mathcal{A} = R(x) \cap x^{\alpha+k}\mathcal{A} \quad \text{for } k \geq 0 .$$

Take $y \in \mathcal{M}$. Then there is a sequence $(u_k)_{k=1}^{\infty}$ in \mathcal{A} such that $y = x^{\alpha+k}u_k$ for $k \geq 1$. Put $z_k = x^{\alpha}u_1 - x^{\alpha+k-1}u_k$. Then $xz_k = 0$, thus $z_k \in R(x) \cap x^{\alpha}\mathcal{A} = R(x) \cap x^{\alpha+k-1}\mathcal{A}$ for all $k \geq 1$. It follows that

$$x^{\alpha}u_1 = z_k + x^{\alpha+k-1}u_k \in x^{\alpha+k-1}\mathcal{A} \quad (k \geq 1).$$

Hence $x^{\alpha}u_1 \in \mathcal{M}$ and therefore $y = x^{\alpha+1}u_1 \in x\mathcal{M}$. The proof of (7.8) is now complete.

(7.8) and the open mapping theorem show that there is a constant $\gamma > 0$ such that

for each $y \in \mathcal{M}$ there is $z \in \mathcal{M}$ with $xz = y$ and $\|z\| \leq \gamma\|y\|$.

Since $R(x) \cap \mathcal{M} = R(x) \cap x^{\alpha}\mathcal{A}$ and $p_l(x) = \infty$, we get some $a_0 \in R(x) \cap \mathcal{M}$ with $a_0 \neq 0$ (Proposition 7.1 (1)).

Now use (7.9) to construct a sequence $(a_n)_{n=1}^{\infty}$ such that

$$xa_{n+1} = a_n \quad \text{and} \quad \|a_n\| \leq \gamma^n \|a_0\| \quad \text{for} \quad n \in \mathbb{N}.$$

Put $U = \{\lambda \in \mathbb{C} : |\lambda| < 1/\gamma\}$ and $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$. Then f is holomorphic on U and a simple computation gives

$$(\lambda e - x)f(\lambda) = -xa_0 = 0 \quad \text{for each} \quad \lambda \in U.$$

From $f(0) = a_0 \neq 0$ we derive $0 \in a_l(x)$.

(ii) \Rightarrow (iii): Since $a_l(x)$ is open, there is $\epsilon > 0$ such that $\lambda \in a_l(x)$ for $|\lambda| < \epsilon$. Take $\lambda_0 \in \mathbb{C}$ with $|\lambda_0| < \epsilon$. Then there is a neighbourhood V of λ_0 and a holomorphic $f : V \rightarrow \mathcal{A}$ with $f(\lambda_0) \neq 0$ and $(\lambda_0 e - x)f(\lambda_0) = 0$. This shows that $R(\lambda_0 e - x) \neq \{0\}$, thus $\text{nul}(\lambda_0 e - x) > 0$.

(iii) \Rightarrow (i) Assume to the contrary that $p_l(x) < \infty$. (1) shows that there is a positive $\delta \leq \epsilon$ such that $\text{nul}(\lambda e - x) = 0$ for $0 < |\lambda| < \delta$, a contradiction. \square

For $x \in \mathcal{A}$ we define

$$\Phi_g(x) = \{\lambda \in \mathbb{C} : \lambda e - x \in \Phi_g(\mathcal{A})\}.$$

It is clear that $\mathbb{C} \setminus \sigma(x) \subseteq \Phi_g(x)$. From Theorem 3.11 (4) we see that $\Phi_g(x)$ is open.

THEOREM 7.10. *Let $x \in \mathcal{A}$ and let C be a connected component of $\Phi_g(x)$.*

- (1) *Either $p_l(\lambda e - x) < \infty$ for all $\lambda \in C$ or $p_l(\lambda e - x) = \infty$ for all $\lambda \in C$.*
- (2) *Either $q_l(\lambda e - x) < \infty$ for all $\lambda \in C$ or $q_l(\lambda e - x) = \infty$ for all $\lambda \in C$.*

PROOF. We only show (1). The proof for (2) is similar since $q_l(\lambda e - x) = p_r(\lambda e - x)$.

Put $M = \{\lambda \in C : p_l(\lambda e - x) < \infty\}$. Theorem 7.7 (1) shows that M is open.

Take $\lambda_0 \in C \setminus M$, hence $p_l(\lambda_0 e - x) = \infty$. Theorem 7.7 (3) gives $\lambda_0 \in a_l(x)$. Since $a_l(x)$ is open, there is $\epsilon > 0$ such that $\lambda \in a_l(x)$ if $|\lambda - \lambda_0| < \epsilon$. Then it follows from Proposition 7.4 (1) that $p_l(\lambda e - x) = \infty$ if $|\lambda - \lambda_0| < \epsilon$. Hence $C \setminus M$ is open. Since C is connected, $M = \emptyset$ or $C = M$. \square

THEOREM 7.11. *For $x \in \Phi_g(\mathcal{A})$ the following assertions are equivalent:*

- (1) *0 is a boundary point of $\sigma(x)$.*
- (2) *0 is an isolated point of $\sigma(x)$.*
- (3) *0 is a pole of $(\lambda e - x)^{-1}$.*

PROOF. The implications (3) \Rightarrow (2) \Rightarrow (1) are clear.

(1) \Rightarrow (3): By C we denote the connected component of $\Phi_g(x)$ for which $0 \in C$. There is some $\epsilon > 0$ such that for $U = \{\lambda \in \mathbb{C} : |\lambda| < \epsilon\}$ we have $U \subseteq C$ and $U \cap (\mathbb{C} \setminus \sigma(x)) \neq \emptyset$. For $\lambda \in U \cap (\mathbb{C} \setminus \sigma(x))$ we have $p_l(\lambda e - x) = q_l(\lambda e - x) = 0$. Theorem 7.10 shows now that $p_l(x), q_l(x) < \infty$. Observe that $p_l(x), q_l(x) > 0$. Proposition 4.1 (1) shows then that $0 < p_l(x) = q_l(x) < \infty$. From [18, Theorem 15.6] (see also [11]) we conclude that 0 is a pole of $(\lambda e - x)^{-1}$. \square

EXAMPLES FOR $\Phi_g(x)$.

(1) If $x \in \text{soc}(\mathcal{A})$ then $\Phi_g(x) = \mathbb{C}$.

(2) If $x \in \overline{\text{soc}(\mathcal{A})} \setminus \text{soc}(\mathcal{A})$ then $\Phi_g(x) = \mathbb{C} \setminus \{0\}$ (see Theorem 6.6).

(3) If x is a Riesz element then $\mathbb{C} \setminus \{0\} \subseteq \Phi_g(x)$.

(4) Let $x \in \mathcal{A}$ with $x^2 = x$. Then $\sigma(x) \subseteq \{0, 1\}$, thus $\mathbb{C} \setminus \{0, 1\} \subseteq \Phi_g(x)$. We also have $(e - x)^2 = e - x$. Therefore, by Example 3.9 (5), $0, 1 \in \Phi_g(x)$. Thus $\Phi_g(x) = \mathbb{C}$.

We close this paper with

THEOREM 7.12. *Suppose that $\dim \mathcal{A} = \infty$ and let $x \in \mathcal{A}$. The following assertions are equivalent:*

(1) $\Phi_g(x) = \mathbb{C}$.

(2) There are $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$ and

$$\prod_{j=1}^m (x - \lambda_j e) \in \text{soc}(\mathcal{A}).$$

PROOF. (1) \Rightarrow (2): Take $\mu \in \sigma(x)$. Since $\mu \in \Phi_g(x)$, it follows from Theorem 3.11 (4) that there is an open neighbourhood U_μ of μ with

$$(7.13) \quad x - \lambda e \in \Phi(\mathcal{A}) \text{ for } \lambda \in U_\mu \setminus \{\mu\}.$$

Since $\sigma(x) \subseteq \bigcup_{\mu \in \sigma(x)} U_\mu$ and $\sigma(x)$ is compact, there are $\lambda_1, \dots, \lambda_n \in \sigma(x)$ such that

$$\sigma(x) \subseteq \bigcup_{j=1}^n U_{\lambda_j}.$$

This and (7.13) show that $\sigma_\Phi(x) = \sigma(\tilde{x}) \subseteq \{\lambda_1, \dots, \lambda_n\}$. Since $\dim \mathcal{A} = \infty$, $\sigma(\tilde{x}) \neq \emptyset$, thus $\sigma(\tilde{x}) = \{\lambda_1, \dots, \lambda_m\}$ with $m \leq n$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Define the polynomial p by $p(\lambda) = \prod_{j=1}^m (\lambda - \lambda_j)$. Then

$$\sigma_\Phi(p(x)) = \sigma(p(\tilde{x})) = \sigma(p(\tilde{x})) = p(\sigma(\tilde{x})) = \{0\}.$$

It follows that $p(x)$ is a Riesz element. Since $x - \lambda_j e \in \Phi_g(\mathcal{A})$ for $j = 1, \dots, m$, we have $p(x) \in \Phi_g(\mathcal{A})$ (see Theorem 5.2 (1)). Now use Theorem 6.7 to get $p(x) \in \text{soc}(\mathcal{A})$.

(2) \Rightarrow (1): Let p denote the polynomial $p(\lambda) = \prod_{j=1}^m (\lambda - \lambda_j)$. Since $p(x) \in \text{soc}(\mathcal{A})$, $\widehat{0} = \widehat{p(x)} = p(\widehat{x})$. Proposition 4.6 yields $\widehat{x} - \lambda \widehat{e} \in \widehat{\mathcal{A}}^g$ for each $\lambda \in \mathbb{C}$, thus $\Phi_g(x) = \mathbb{C}$. \square

REFERENCES

- [1] J. C. ALEXANDER: *Compact Banach algebras*, Proc. London Math. Soc. III, Ser. 18 (1968), 1-18.
- [2] B. A. BARNES: *A generalized Fredholm theory for certain maps in the regular representation of an algebra*, Can. J. Math., **20** (1968), 495-504.
- [3] B. A. BARNES: *The Fredholm elements of a ring*, Can. J. Math., **21** (1969), 84-95.
- [4] B. A. BARNES – G. J. MURPHY – M. R. F. SMYTH – T. T. WEST: *Riesz and Fredholm theory in Banach algebras*, London. Pitman, 1982.
- [5] F. F. BONSAALL – J. DUNCAN: *Complete normed algebras*, Berlin. Springer, 1973.
- [6] S. R. CARADUS: *Operator theory of the pseudo-inverse*, Queen's papers in pure and appl. math. No. 38 (1974).
- [7] S. R. CARADUS: *Generalized inverses and operator theory*, Queen's papers in pure and appl. math. No. 50 (1978).
- [8] H. HEUSER: *Funktionalanalysis*, 2. Aufl. Stuttgart. Teubner, 1986.
- [9] D. C. LAY: *Spectral analysis using ascent, descent nullity and defect*, Math. Ann., **184** (1971), 197-214.
- [10] D. MÄNNLE: *Verallgemeinerte Fredholmelemente in halbeinfachen Banachalgebren*, Diplomarbeit, Karlsruhe, 1998.
- [11] L. D. PEARLMAN: *Riesz points of the spectrum of an element in a semisimple Banach algebra*, Trans. Amer. Math. Soc., **193** (1974), 303-328.
- [12] CH. SCHMOEGER: *Atkinson theory an holomorphic functions in Banach algebras*, Proc. R. Ir. Acad. 91A, No. 1 (1991), 113-127.
- [13] CH. SCHMOEGER: *Ascent, descent, and the Atkinson region in Banach algebras*, Ricerche Mat., **42** (1993), 123-143.

-
- [14] CH. SCHMOEGER: *Ascent, descent, and the Atkinson region in Banach algebras*, II. *Ricerche Mat.*, **42** (1993), 249-264.
- [15] CH. SCHMOEGER: *On a class of generalized Fredholm operators*, I. *Demonstratio Math.*, **30** (1997), 829-842.
- [16] CH. SCHMOEGER: *On a class of generalized Fredholm operators*, II. *Demonstratio Math.*, **31** (1998), 705-722.
- [17] CH. SCHMOEGER: *On a class of generalized Fredholm operators*, III. *Demonstratio Math.*, **31** (1998), 723-733.
- [18] M. SCHREIECK: *Atkinson-, Fredholm- und Rieszelemente*, Dissertation, Karlsruhe, 1984.
- [19] M. R. F. SMYTH: *Fredholm theory in Banach algebras*, Banach center publications, **8** (1982), 403-414.
- [20] Ä. WECKBACH: *Wesentliche Spektren und Störungen von Atkinson- und Fredholm-Elementen*, Dissertation, Karlsruhe, 1983.

*Lavoro pervenuto alla redazione il 18 gennaio 1999
ed accettato per la pubblicazione il 23 febbraio 2000.
Bozze licenziate il 10 maggio 2000*

INDIRIZZO DEGLI AUTORI:

Daniel Männle – Christoph Schmoeger – Math. Institut I – Universität Karlsruhe – D-76128
Karlsruhe, Germany
E-mail: christoph.schmoeger@math.uni-karlsruhe.de