GENERALIZED PROJECTIONS IN BANACH ALGEBRAS

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Abstract. In this note we investigate generalized projections in Banach algebras. Our results generalize results obtained for bounded linear operators on Hilbert spaces.

1. Introduction and terminology

A bounded linear operator on a Hilbert space is said to be a *generalized projection* if

$$A^2 = A^* .$$

For properties and characterizations of generalized projection operators see [3], [4], and [5] (and the references there). The aim of this note are generalizations of some of the results obtained in [3] and [4] to elements of complex Banach algebras.

Throughout this paper $\mathscr{A}$ will denote a complex unital Banach algebra with unit $\mathbf{1}$.

If $a \in \mathscr{A}$, then we denote the spectrum and the spectral radius of a by $\sigma(a)$ and $r(a)$, respectively.

An element $h \in \mathscr{A}$ is said to be *hermitian* if $\| \exp(ith) \| = 1$ for all $t \in \mathbb{R}$. $\mathscr{H}(\mathscr{A})$ denotes the set of hermitian elements of $\mathscr{A}$. It is well-known that a bounded linear operator H on a complex Hilbert space is hermitian if and only if $H = H^*$.

We collect the basic properties of $\mathscr{H}(\mathscr{A})$ (for proofs see [2]):

1.1. Proposition.
(1) *$\mathscr{H}(\mathscr{A})$ is closed real subspace of $\mathscr{A}$ and $\mathscr{H}(\mathscr{A}) \cap i\mathscr{H}(\mathscr{A}) = \{0\}$.*
(2) *If $h, k \in \mathscr{H}(\mathscr{A})$, then*

$$i(hk - kh) \in \mathscr{H}(\mathscr{A}),\ \sigma(h) \subseteq \mathbb{R} \ \text{ and } \ r(h) = \|h\| .$$

The following example (due to M. J. Crabb, see [2, page 57]) shows that, in contrast to the operator situation, if $h \in \mathscr{H}(\mathscr{A})$, then it does not follow that $h^2 \in \mathscr{H}(\mathscr{A})$.

1.2. Example. *Let $\mathscr{A} = \mathbb{C}^3$ with pointwise multiplication and let $p : \mathscr{A} \to [0, \infty)$ be defined by*

$$p(\alpha, \beta, \gamma) = \sup \{ |\lambda^{-1}\alpha + \beta + \lambda\gamma| : \lambda \in \mathbb{C},\ |\lambda| = 1 \} .$$

Define the norm $\| \cdot \|$ on $\mathscr{A}$ by

$$\|a\| = \sup \{ p(xa) : x \in \mathscr{A},\ p(x) = 1 \} .$$

Then $(\mathscr{A}, \| \cdot \|)$ is a complex (commutative) Banach algebra with unit $\mathbf{1}= (1, 1, 1)$. Let $h_0 = (-1, 0, 1)$, then

$$h_0 \in \mathscr{H}(\mathscr{A}),\ h_0^2 \notin \mathscr{H}(\mathscr{A}) .$$

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Let $\mathscr{J}(\mathscr{A}) = \{h + ik : h,\, k \in \mathscr{H}(\mathscr{A})\}$. Since $\mathscr{H}(\mathscr{A}) \cap i\mathscr{H}(\mathscr{A}) = \{0\}$, each element of $\mathscr{J}(\mathscr{A})$ has a *unique* representation of the form $h + ik$ with $h,\, k \in \mathscr{H}(\mathscr{A})$. Therefore we may define a linear involution * on $\mathscr{J}(\mathscr{A})$ by

$$(h + ik)^* = h - ik\,.$$

We say that $a \in \mathscr{J}(\mathscr{A})$ is *normal* if $aa^* = a^*a$. If $a = h + ik \in \mathscr{J}(\mathscr{A})\,(h,\, k \in \mathscr{H}(\mathscr{A}))$, then it is easy to see that a is normal if and only if $hk = kh$.

If $\mathscr{S}$ is a commutative subset of $\mathscr{A}$, then the *centralizer* of $\mathscr{S}$ is given by

$$\Gamma(\mathscr{S}) = \{x \in \mathscr{A} : xs = sx \quad \text{for every } s \in \mathscr{S}\}\,.$$

We have (see [7, 11.21])

$$\mathscr{S} \subseteq \Gamma(\Gamma(\mathscr{S}))$$

and $\Gamma(\Gamma(\mathscr{S}))$ *is a commutative Banach algebra* (with unit $\mathbf{1}$).

Notation. For a normal element $a \in \mathscr{J}(\mathscr{A})$, $a = h + ik$ with $h,\, k \in \mathscr{H}(\mathscr{A})$, let

$$\mathscr{B}(a) = \Gamma(\Gamma(\{h, k\}))$$

and let Δ_a denote the set of all nontrivial complex homomorphisms of $\mathscr{B}(a)$.

1.3. Proposition. *Suppose that* $a = h + ik \in \mathscr{J}(\mathscr{A})\,(h, k \in \mathscr{H}(\mathscr{A}))$ *is normal. Then*

(1) $\sigma(x) = \{\varphi(x) : \varphi \in \Delta_a\}$ *for all* $x \in \mathscr{B}(a)$*;*

(2) $\varphi(h),\, \varphi(k) \in \mathbb{R}$ *for all* $\varphi \in \Delta_a$*;*

(3) $\varphi(a^*) = \overline{\varphi(a)}$ *for all* $\varphi \in \Delta_a$*, and* $\sigma(a^*) = \{\overline{\lambda} : \lambda \in \sigma(a)\}$*;*

(4) *if* $x,\, y \in \mathscr{B}(a)$*, then*

$$\sigma(x + y) \subseteq \sigma(x) + \sigma(y),\ \sigma(xy) \subseteq \sigma(x)\sigma(y)\,.$$

Proof. (1) [7, Theorem 11.9, Theorem 11.22].

(2) Follows from (1) and Proposition 1.1 (2).

(3) $\varphi(a^*) = \varphi(h - ik) = \varphi(h) - i\varphi(k) = \overline{\varphi(h) + i\varphi(k)} = \overline{\varphi(a)}$.

(4) [7, Theorem 11.23]. □

2. Generalized projections in Banach algebras

An element $a \in \mathscr{J}(\mathscr{A})$ is called a *generalized projection* if $a^2 = a^*$. We say that $a \in \mathscr{J}(\mathscr{A})$ is a *partial isometry* if $a = aa^*a$.

2.1. Theorem. *Suppose that* $a = h + ik \in \mathscr{J}(\mathscr{A})$ *is a generalized projection. Then we have:*

(1) *a is normal;*

(2) $\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = 1\}$*;*

(3) *if, in addition,* $hk \in \mathscr{H}(\mathscr{A})$*, then*

$$(a^*)^2 = a \quad \text{and} \quad a^4 = a\,.$$

Remark. If $\mathscr{A}$ is the Banach algebra of all bounded linear operators on a complex Hilbert space, then the condition $hk \in \mathscr{H}(\mathscr{A})$ in part (3) of the above theorem can be dropped, since a is normal.

Proof of Theorem 2.1. (1) Since $a^2 = a^*$,

$$aa^* = aa^2 = a^2a = a^*a\,.$$

(2) Let $\lambda \in \sigma(a)$. Then $\lambda = \varphi(a)$ for some $\varphi \in \Delta_a$, hence, by Proposition 1.3,

$$\overline{\lambda} = \varphi(a^*) = \varphi(a^2) = \varphi(a)^2 = \lambda^2\,.$$

If $\lambda \neq 0$, then $|\lambda| = 1$ and $\lambda^3 = \lambda\lambda^2 = \lambda\overline{\lambda} = 1$.
(3) From $a^2 = a^*$ we see that

$$h^2 - h - k^2 = i(-k - 2hk)\,.$$

Proposition 1.3 (4) and Proposition 1.1 (2) show that

$$\sigma(h^2 - h - k^2),\ \sigma(-k - 2hk) \subseteq \mathbb{R}\,,$$

hence

$$\sigma(-k - 2hk) \subseteq \mathbb{R} \cap i\mathbb{R} = \{0\}\,,$$

thus $r(-k - 2hk) = 0$. Since $hk \in \mathscr{H}(\mathscr{A})$, $-k - 2hk \in \mathscr{H}(\mathscr{A})$. Now use Proposition 1.1 (2) to get $k = -2hk$, hence $h^2 - k^2 = h$. Therefore

$$(a^*)^2 = h^2 - 2ihk - k^2 = h - 2ihk = h - (-ik) = h + ik = a$$

and

$$a^4 = (a^*)^2 = a\,.$$

□

2.2. Theorem. *Let $a = h + ik \in \mathscr{J}(\mathscr{A})$ and suppose that*

$$\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = 1\}\,.$$

We have:
(1) *if a is normal, then $r(a^2 - a^*) = 0$;*
(2) *if $hk, h^2, k^2 \in \mathscr{H}(\mathscr{A})$, then a is a generalized projection.*

Before we give a proof of the above theorem we have the following corollary, which generalizes Theorem 2 in [3] and Theorem 1 in [4] (see Remark (2) below).

2.3. Corollary. *For $a = h + ik \in \mathscr{J}(\mathscr{A})$ with $h, k, hk, h^2, k^2 \in \mathscr{H}(\mathscr{A})$ the following assertions are equivalent:*
(1) *a is a normal partial isometry and $a^4 = a$;*
(2) *a is normal and $a^4 = a$;*
(3) *$\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = 1\}$;*
(4) *$a^2 = a^*$.*

Proof. (1) ⇒ (2): Clear.
(2) ⇒ (3): Use the spectral mapping theorem ([7, Theorem 10.28]).
(3) ⇒ (4): Theorem 2.2 (2).
(4) ⇒ (1): By Theorem 2.1 (1), a is normal, and part (3) of Theorem 2.1 gives $aa^*a = aa^2a = a^4 = a$. □

Remarks. (1) The implication (1) ⇒ (2) ⇒ (3) in Corollary 2.3 are valid without the assumptions that $hk, h^2, k^2 \in \mathscr{H}(\mathscr{A})$.
(2) If $\mathscr{A}$ is the Banach algebra of all bounded linear operators on a complex Hilbert space, then the condition $hk \in \mathscr{H}(\mathscr{A})$ is equivalent to the normality of a, thus the assumptions $h^2, k^2 \in \mathscr{H}(\mathscr{A})$ in Theorem 2.2 (2) and Corollary 2.3 can be dropped in the operator situation.

Proof of Theorem 2.2. Let $b = a^2 - a^*$.
(1) Since a is normal, $hk = kh$, thus $b \in \mathscr{B}(a)$. Take $\lambda \in \sigma(b)$. Then $\lambda = \varphi(b) = \varphi(a)^2 - \overline{\varphi(a)}$ for some $\varphi \in \Delta_a$.
Case 1: $\varphi(a) = 0$. Then we have $\lambda = 0$.
Case 2: $\varphi(a) \neq 0$. Since $\varphi(a) \in \sigma(a)$, $\varphi(a)^3 = 1$.
It follows that

$$\lambda\varphi(a) = \varphi(a)^3 - \overline{\varphi(a)}\varphi(a) = 1 - 1 = 0\,,$$

and so $\lambda = 0$. Hence $\sigma(b) = \{0\}$.
(2) Since h, k, hk, h^2 and k^2 are all hermitian, it follows from [1, Theorem 2.14] (see also [6]) that $hk = kh$. Thus a is normal. From

$$b = h^2 + 2ihk - k^2 - (h - ik)$$

we get

$$i(2hk + k) = b - (h^2 - h - k^2)\,.$$

By Proposition 1.1 and Proposition 1.3,

$$\sigma(2hk + k) \subseteq \mathbb{R} \cap i\mathbb{R} = \{0\}\,.$$

Since $2hk + k \in \mathscr{H}(\mathscr{A})$, $2hk + k = 0$, thus

$$b = h^2 - h - k^2 \in \mathscr{H}(\mathscr{A})\,.$$

Since, by (1), $r(b) = 0$, we conclude that $b = 0$, hence $a^2 = a^*$. □

2.4. Example. *Let $\mathscr{A}$ and h_0 as in Example 1.2. Observe that*

$$\sigma((\alpha, \beta, \gamma)) = \{\alpha, \beta, \gamma\} \qquad ((\alpha, \beta, \gamma) \in \mathscr{A})\,.$$

Hence

$$\sigma(a) = \{0\} \Leftrightarrow a = 0 \qquad (a \in \mathscr{A})\,. \tag{2.1}$$

The following properties are shown in [2]:

$$\mathscr{A} = \{\alpha\mathbf{1} + \beta h_0 + \gamma h_0^2 : \alpha, \beta, \gamma \in \mathbb{C}\}\,,$$

$$\mathscr{H}(\mathscr{A}) = \{\alpha\mathbf{1} + \beta h_0 : \alpha, \beta \in \mathbb{R}\}\,. \tag{2.2}$$

Hence we have

$$\mathscr{J}(\mathscr{A}) = \{\xi\mathbf{1} + \eta h_0 : \xi, \eta \in \mathbb{C}\}\,,$$

and each element of $\mathscr{J}(\mathscr{A})$ is normal. Furthermore, if

$$a = \xi\mathbf{1} + \eta h_0 \in \mathscr{J}(\mathscr{A}),\ \text{then } a^* = \overline{\xi}\mathbf{1} + \overline{\eta} h_0\,.$$

For $a \in \mathscr{J}(\mathscr{A})$ the following assertions are equivalent:

$$\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = 1\}\,; \tag{2.3}$$

$$a^2 = a^*\,; \tag{2.4}$$

$$a \in \{(\lambda, \lambda, \lambda) : \lambda = 0 \text{ or } \lambda^3 = 1\}\,; \tag{2.5}$$

$$a^4 = a\,. \tag{2.6}$$

Proof. (2.3) ⇒ (2.4): Since a is normal, we have $\sigma(a^2 - a^*) = \{0\}$ (Theorem 2.2), thus, by (2.1), $a^2 = a^*$.
(2.4) ⇒ (2.5): Let $a = \xi\mathbf{1} + \eta h_0\,(\xi, \eta \in \mathbb{C})$. Since $a^2 = a^*$,

$$((\xi - \eta)^2, \xi^2, (\xi + \eta)^2) = (\overline{\xi} - \overline{\eta}, \overline{\xi}, \overline{\xi} + \overline{\eta})\,.$$

Case 1: $\xi \neq 0$. Then $\eta^2 = -\overline{\eta}$ and $\eta^2 = \overline{\eta}$, thus $\xi = \eta = 0$, hence $a = (0, 0, 0)$.
Case 2: $\xi \neq 0$. Since $\xi^2 = \overline{\xi}$,

$$\overline{\xi} - \overline{\eta} = (\xi - \eta)^2 = \xi^2 - 2\xi\eta + \eta^2 = \overline{\xi} - 2\xi\eta + \eta^2$$

and

$$\overline{\xi} + \overline{\eta} = [\xi + \eta)^2 = \xi^2 + 2\xi\eta + \eta^2 = \overline{\xi} + 2\xi\eta + \eta^2\,,$$

hence

$$\eta^2 = 2\xi\eta - \overline{\eta} = -\eta^2\,,$$

therefore $\eta = 0$. From $\xi^2 = \overline{\xi}$ it follows that $|\xi| = 1$ and $\xi^3 = \xi\overline{\xi} = 1$.
(2.5) ⇒ (2.6): Clear.
(2.6) ⇒ (2.3): Use the spectral mapping theorem ([7, Theorem 10.28]). ☐

Our next results generalize results obtained in [4] for complex $n \times n$ matrices.

2.5. Theorem. *Let $a, b \in \mathscr{J}(\mathscr{A})$ be generalized projections. Then $a + b$ is a generalized projection if and only if $ab = ba = 0$.*

Proof. We have

$$(a + b)^2 = a^2 + ab + ba + b^2 = a^* + ab + ba + b^*\,.$$

If $ab = ba = 0$, then $(a + b)^2 = a^* + b^* = (a + b)^*$, hence $a + b$ is a generalized projection. Conversely, assume that $(a + b)^2 = (a + b)^*$. Then $ab + ba = 0$. Applying a on the left and on the right gives

$$a^*b + aba = 0 = aba + ba^*\,,$$

hence $a^*b = ba^*$. Since a is normal, a^* is normal. Proposition 2.1 in [8] implies now that $ab = ba$, thus $ab = ba = 0$. ☐

2.6. Theorem. *Let $a, b \in \mathscr{J}(\mathscr{A})$ be generalized projections, $a = h + ik$ $(h, k \in \mathscr{H}(\mathscr{A}))$. If in addition $hk \in \mathscr{H}(\mathscr{A})$, then*

$$(b - a)^2 = (b - a)^* \Leftrightarrow ab = ba = a^*\,.$$

Proof. By Theorem 2.1 (3), $(a^*)^2 = a$ and $a^4 = a$. First assume that $ab = ba = a^*$. Then

$$(b - a)^2 = b^2 - ba - ab + a^2 = b^* - a^* = (b - a)^*\,,$$

hence $b - a$ is a generalized projection.
Now assume that $(b - a)^2 = (b - a)^*$. It follows that

$$2a^* = ab + ba\,. \tag{2.7}$$

Multiplying (2.7) from the left by a^* gives

$$2a = 2(a^*)^2 = a^*ab + a^*ba\,.$$

Multiplying (2.7) from the right by a^* gives

$$2a = 2(a^*)^2 = aba^* + baa^* .$$

Hence, since $aa^* = a^3 = a^*a$,

$$a^3b + a^*ba = aba^2 + ba^3 . \tag{2.8}$$

Multiplying (2.8) from the right by a yields

$$a^3ba + a^*ba^* = aba^*a + ba ,$$

thus

$$ba = a^3ba + a^*ba^* - aba^3 . \tag{2.9}$$

Similar arguments show that

$$ab = aba^3 + a^*ba^* - a^3ba , \tag{2.10}$$

thus

$$ab + ba = 2a^*ba^* . \tag{2.11}$$

Now use (2.7) to get

$$a^2 = a^*ba^* . \tag{2.12}$$

From (2.12) we see that

$$aba = (a^*)^2b(a^*)^2 = a^*(a^*ba^*)a^* = (a^*)^3 ,$$

hence

$$aba = a^6 = a^4a^2 = a^3 = a^*a = aa^* , \tag{2.13}$$

thus

$$a^*ba = a^2ba = aa^*a - a^4 = a , \tag{2.14}$$

and

$$aba^* = aba^2 = a^*a^2 = a^4 = a . \tag{2.15}$$

Then (2.9)–(2.15) show that $ab = a^* = ba$. ☐

3. q-generalized projections

If $q \in \mathbb{N}$ and $q > 1$, a *q-generalized projection* is an element $a \in \mathscr{J}(\mathscr{A})$ such that $a^q = a^*$ (q-generalized projection operators on complex Hilbert spaces are considered in [5]).

With obvious modifications of the proofs in Section 2, we see that the following results a true:

3.1. Theorem. *If $a \in \mathscr{J}(\mathscr{A})$ is a q-generalized projection, then a is normal and*

$$\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C}, \lambda^{q+1} = 1\} .$$

3.2. Theorem. *Let $a = h + ik \in \mathscr{J}(\mathscr{A})(h, k \in \mathscr{H}(\mathscr{A}))$ and suppose that*

$$\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^{q+1} = 1\} .$$

(1) *If a is normal, then $r(a^q - a^*) = 0$.*
(2) *If $h^nk^m \in \mathscr{H}(\mathscr{A})$ for $n, m \in \{0, 1, \ldots, q\}$ then a is normal and $a^q = a^*$.*

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