ON PSEUDO-INVERSES OF FREDHOLM OPERATORS

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Abstract. Suppose that A is a Fredholm operator on a Banach space. We prove that A has index $= 0$ (resp. ≥ 0, resp. ≤ 0) if and only if A has pseudo-inverse which is invertible (resp. Fredholm and left invertible, resp. Fredholm and right invertible). Furthermore we determine the interior points of some classes of linear operators.

1. Terminology

X always denotes a complex Banach space, and the algebra of all bounded linear operators on X is denoted by $\mathscr{L}(X)$.
If $A \in \mathscr{L}(X)$ we denote by $N(A)$ the kernel of A and by $\alpha(A)$ the dimension of $N(A)$. $A(X)$ denotes the range of A, and we define $\beta(A) = \operatorname{codim} A(X)$.

An operator $A \in \mathscr{L}(X)$ is called *relatively regular* if there is $S \in \mathscr{L}(X)$ such that $ASA = A$. In this case S is called a *pseudo-inverse* of A, and, if $B = SAS$, then

$$ABA = A \quad \text{and} \quad BAB = B\,.$$

$A \in \mathscr{L}(X)$ is called a *Fredholm operator* if $\alpha(A)$ and $\beta(A)$ are both finite. In this case we define the *index* of A by $\operatorname{ind}(A) = \alpha(A) - \beta(A)$.
Observe that a Fredholm operator is relatively regular [2, Satz 74.4].

Let $A \in \mathscr{L}(X)$. The sequence $N(A)$, $N(A^2)$, $N(A^3)$, $\ldots$ is increasing, while the sequence $A(X)$, $A^2(X)$, $A^3(X)$, $\ldots$ is decreasing. Define $p(A)$, the *ascent* of A, to be the smallest integer $p \geq 0$ such that $N(A^p) = N(A^{p+1})$ or ∞ if no such p exists.
Define $q(A)$, the *descent* of A, to be the smallest integer $q \geq 0$ with $A^q(X) = A^{q+1}(X)$ or ∞ if no such q exists. It is shown in [2, Satz 72.3], that if $p(A) < \infty$ and $q(A) < \infty$, then $p(A) = q(A)$.

We define various classes of operators:

$\Phi(X) = \{A \in \mathscr{L}(X) : A \text{ is Fredholm}\}$;

$\Phi_\alpha(X) = \{A \in \Phi(X) : \alpha(A) = 0\}$;

$\Phi_\beta(X) = \{A \in \Phi(X) : \beta(A) = 0\}$;

$\mathscr{L}(X)^{-1} = \{A \in \mathscr{L}(X) : \alpha(A) = \beta(A) = 0\}$;

$\mathscr{F}(X) = \{A \in \mathscr{L}(X) : \dim A(X) < \infty\}$.

Since Fredholm operators are relatively regular, $\Phi_\alpha(X)$ is the set of all left invertible Fredholm operators and $\Phi_\beta(X)$ is the class of all right invertible Fredholm operators.

The main results of this paper read as follows:

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1.1. Theorem. *If* $A \in \Phi(X)$, *then*
(a) $\operatorname{ind}(A) = 0 \Leftrightarrow$ *there is* $S \in \mathscr{L}(X)^{-1}$ *such that* $ASA = A$;
(b) $\operatorname{ind}(A) \geq 0 \Leftrightarrow$ *there is* $S \in \Phi_\alpha(X)$ *such that* $ASA = A$;
(c) $\operatorname{ind}(A) \leq 0 \Leftrightarrow$ *there is* $S \in \Phi_\beta(X)$ *such that* $ASA = A$.

1.2. Theorem. *If* $A \in \Phi(X)$, *then* $p(A) = q(A) < \infty$ *if and only if there are* $p \in \mathbb{N}_0$ *and* $S \in \mathscr{L}(X)^{-1}$ *such that* $A^pSA^p = A^p$ *and* $A^pS = SA^p$.

Proofs of the above result will be given in the next section.

2. Proofs

2.1. Proposition. *Suppose that* $A \in \mathscr{L}(X)$ *is relatively regular and* B *is a pseudo-inverse of* A *with* $ABA = A$ *and* $BAB = B$.
(a) AB, BA, $I - AB$ *and* $I - BA$ *are projections with*

$$(AB)(X) = A(X),\ (BA)(X) = B(X)$$
$$(I - AB)(X) = N(B) \quad \text{and} \quad (I - BA)(X) = N(A)\,.$$

(b) *If* $A \in \Phi(X)$, *then* $B \in \Phi(X)$, $\alpha(B) = \beta(A)$, $\beta(B) = \alpha(A)$ *and* $\operatorname{ind}(B) = -\operatorname{ind}(A)$.

Proof. Easy verification. □

2.2. Proposition. *Let* A *and* B *as in Proposition 2.1 and suppose that* $A \in \Phi(X)$. *Then there are* $R \in \Phi(X)$ *and* $F \in \mathscr{F}(X)$ *such that*

$$BF = 0 \quad \text{and} \quad A = R + F\,.$$

Furthermore we have:
(a) *if* $\operatorname{ind}(A) = 0$, *then* $R \in \mathscr{L}(X)^{-1}$;
(b) *if* $\operatorname{ind}(A) \geq 0$, *then* $R \in \Phi_\beta(X)$;
(c) *if* $\operatorname{ind}(A) \leq 0$, *then* $R \in \Phi_\alpha(X)$.

Proof. By Proposition 2.1, $(AB)(X) = A(X)$ and $(I - AB)(X) = N(B)$. Hence

$$X = A(X) \oplus N(B)\,.$$

Since $\alpha(A) < \infty$, there is $P \in \mathscr{L}(X)$ such that $P^2 = P$ and $P(X) = N(A)$. Let $n = \alpha(A)$, $m = \beta(A)$ and $p = \min\{n, m\}$. Let $\{x_1, \dots, x_n\}$ be a basis of $N(A)$. Then there are $x_1^*, \dots, x_n^* \in X^*$ linearly independent with

$$Px = \sum_{j=1}^{n} x_j^*(x) x_j \qquad (x \in X)\,.$$

If $\{y_1, \dots, y_m\}$ is a basis of $N(B)$, define $F \in \mathscr{F}(X)$ by

$$Fx = \sum_{j=1}^{p} x_j^*(x) y_j \qquad (x \in X)\,.$$

Then $F(X) \subseteq N(B)$, thus $BF = 0$. Let $R = A - F$. It is shown in the proof of Satz 77.2 in [2] that (a), (b) and (c) hold. □

Proof of Theorem 1.1. Let B, F and R as in Proposition 2.2. Then $BA = BR + BF = BR$, hence $A = ABA = ABR$

(a) $\Rightarrow$: Since $\mathrm{ind}\,(A) = 0$, we have $R \in \mathscr{L}(X)^{-1}$. Thus $AR^{-1} = AB$, hence $A = ABA = AR^{-1}A$.
$\Leftarrow$: By the Index-theorem([2, Satz 71.3]),

$$\mathrm{ind}\,(A) = 2\,\mathrm{ind}\,(A) + \mathrm{ind}\,(S) = 2\,\mathrm{ind}\,(A)\,,$$

hence $\mathrm{ind}\,(A) = 0$.

(b) $\Rightarrow$: Since $\mathrm{ind}\,(A) \geq 0$, there is $S \in \mathscr{L}(X)$ such that $RS = I$. From $R \in \Phi_\beta(X)$ we get, by Proposition 2.1 (b), $S \in \Phi_\alpha(X)$. From $A = ABR$ it results that $AS = AB$, hence $A = ABA = ASA$.
$\Leftarrow$: Use the Index-theorem to see that $\mathrm{ind}\,(A) \geq 0$.

(c) $\Rightarrow$: Since $\mathrm{ind}\,(A) \leq 0$, we have $\mathrm{ind}\,(B) \geq 0$, by Proposition 2.1(b). Apply Proposition 2.2 to B. Hence there are $F_0 \in \mathscr{F}(X)$, $R_0 \in \Phi_\beta(X)$ such that

$$AF_0 = 0 \quad \text{and} \quad B = R_0 + F_0\,.$$

Let $S = R_0$. From $AB = AR_0 + AF_0 = AS$, we derive $A = ABA = ASA$.
$\Leftarrow$: We have $\mathrm{ind}\,(A) \leq 0$, by the Index-theorem. $\square$

Proof of Theorem 1.2.
(a) $\Rightarrow$ (b): Let $p = p(A) = q(A) < \infty$. Satz 72.4 in [2] gives

$$X = N(A^p) \oplus A^p(X)\,.$$

From [2, Satz 101.2] we see that 0 is a pole if the resolvent $(\lambda I - A)^{-1}$. Let P be the associated spectral projection. Hence

$$P(X) = N(A^p) \quad \text{and} \quad N(P) = A^p(X)\,.$$

Then $PA = AP$ by [2, Satz 99.1]. Let $F = AP + P$ and $R = A(I - P) - P$. Then $A = R + F$. The proof of Satz 77.4 in [2] shows that R is invertible in $\mathscr{L}(X)$. Furthermore we have $RF = FR$, $AF = FA$ and $AR = RA$. Since $F(X) \subseteq P(X) = N(A^p)$, we get $A^pF = 0$, thus $A^{p+1} = A^pR$.
Case 1: $p = 0$. With $S = I$ we are done.
Case 2: $p = 1$. We have $A^2 = AR$. Let $S = R^{-1}$. Then $AS = SA$ and $A = A^2S = ASA$.
Case 3: $p > 1$. Let $A_0 = A^p$. Satz 71.2 in [2] shows that A_0 is a Fredholm operator. From

$$N(A_0^2) = N(A^{2p}) = N(A^p) = N(A_0)\,,$$

and

$$A_0^2(X) = A^{2p}(X) = A^p(X) = A_0(X)\,,$$

we conclude that $p(A_0) = q(A_0) \leq 1$. Case 1 and Case 2 show that there is an invertible operator S in $\mathscr{L}(X)$ with $A_0S = SA_0$ and $A_0 = A_0SA_0$.

(b) $\Rightarrow$ (a): Assume that $p \in \mathbb{N}_0$, $S \in \mathscr{L}(X)$ is invertible, $A^pS = SA^p$ and $A^p = A^pSA^p$. Then $A^{2p}S = A^p$. It follows that $A^p(X) = A^{2p}(S(X)) = A^{2p}(X)$, thus $q(A) < \infty$. Furthermore $N(A^{2p}) = N(A^p)$, hence $p(A) < \infty$. $\square$

3. Interior points of some classes of operators

For a subset $\mathscr{M}$ of $\mathscr{L}(X)$ let $\mathrm{cl}\,(\mathscr{M})$ and $\mathrm{int}\,(\mathscr{M})$ denote the closure and the interior of $\mathscr{M}$, respectively.

Notation.

$\Phi_+(X) = \{A \in \Phi(X) : \mathrm{ind}\,(A) \geq 0\}$;

$\Phi_-(X) = \{A \in \Phi(X) : \mathrm{ind}\,(A) \leq 0\}$;

$\Phi_0(X) = \{A \in \Phi(X) : \mathrm{ind}\,(A) = 0\}$;

$\mathscr{R}(X) = \{A \in \mathscr{L}(X) : A \text{ is relatively regular}\}$;

$\mathscr{A}(X) = \{A \in \mathscr{R}(X) : \alpha(A) < \infty \text{ or } \beta(A) < \infty\}$;

$\mathscr{R}_\alpha(X) = \{A \in \mathscr{R}(X) : ABA = A \text{ for some } B \in \Phi_\alpha(X)\}$;

$\mathscr{R}_\beta(X) = \{A \in \mathscr{R}(X) : ABA = A \text{ for some } B \in \Phi_\beta(X)\}$;

$\mathscr{R}_0(X) = \{A \in \mathscr{R}(X) : ABA = A \text{ for some } B \in \mathscr{L}(X)^{-1}\}$.

Operators of the class $\mathscr{A}$(X) are called *Atkinson operators* or *relatively regular semi-Fredholm operators.*

3.1. Proposition.
(a) $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_0(X)$ *and* $\mathscr{A}(X)$ *are open subsets of* $\mathscr{L}(X)$.
(b) $\mathscr{R}_\alpha(X) \cup \mathscr{R}_\beta(X) \subseteq \mathrm{cl}\,(\Phi(X))$.

Proof. (a) follows from [2, Satz 82.4] and (b) is shown in [3, Theorem 3]. ☐

From Proposition 3.1 (a) and Theorem 1.1 we get

$$\Phi_0(X) \subseteq \mathrm{int}\,(\mathscr{R}_0(X)),\ \Phi_+(X) \subseteq \mathrm{int}\,(\mathscr{R}_\alpha(X)),$$
$$\Phi_-(X) \subseteq \mathrm{int}\,(\mathscr{R}_\beta(X)) \quad \text{and} \quad \mathscr{A}(X) \subseteq \mathrm{int}\,(\mathscr{R}(X)).$$

We can be more precise:

3.2. Theorem.
(a) $\mathrm{int}\,(\mathscr{R}(X)) = \mathscr{A}(X)$;
(b) $\mathrm{int}\,(\mathscr{R}_0(X)) = \Phi_0(X)$;
(c) $\mathrm{int}\,(\mathscr{R}_\alpha(X)) = \Phi_+(X)$;
(d) $\mathrm{int}\,(\mathscr{R}_\beta(X)) = \Phi_-(X)$.

Proof. We only have to show the inclusion "$\subseteq$".
(a) Let $A \in \mathrm{int}\,(\mathscr{R}(X))$. Suppose that $A \notin \mathscr{A}(X)$. Then $\alpha(A) = \beta(A) = \infty$. From [1, Theorem V. 2.6] we know that there is a compact $K \in \mathscr{L}(X)$ such that the range $(A+\lambda K)(X)$ is not closed for all $\lambda \in \mathbb{C}\setminus\{0\}$. Since A is an interior point of $\mathscr{R}(X)$, $A+\lambda K$ has closed range for $|\lambda|$ sufficiently small, a contradiction.
(b), (c) and (d) Let $\gamma \in \{0, \alpha, \beta\}$ and $A \in \mathrm{int}\,(\mathscr{R}_\gamma(X))$. Hence $A \in \mathrm{int}\,(\mathscr{R}(X))$, thus $A \in \mathscr{A}(X)$, by (a). Proposition 3.1 (b) shows that there is a sequence (A_n) in $\Phi(X)$ such that $\|A_n - A\| \to 0\,(n \to \infty)$. Since $\mathscr{A}(X)$ is open, the stability of the index ([2, Satz 82.4]) shows that

$$\mathrm{ind}\,(A_n) = \mathrm{ind}\,(A) \text{ for } n \text{ sufficiently large}.$$

Thus $\mathrm{ind}\,(A)$ is finite, and so $A \in \Phi(X)$. Since $A \in \mathscr{R}_\gamma(X)$, Theorem 1.1 completes the proof. ☐

3.3. Theorem.

$$\Phi_\alpha(X) \cup \Phi_\beta(X) = \operatorname{int}(\{A \in \Phi(X) : N(A) \subseteq A(X)\}).$$

Proof. Since $\Phi_\alpha(X)$ and $\Phi_\beta(X)$ are open, the inclusion "$\subseteq$" is clear. Now suppose that $A \in \operatorname{int}(\{A \in \Phi(X) : N(A) \subseteq A(X)\})$. Then there is $\epsilon > 0$ such that

$(*)$ if $B \in \mathscr{L}(X)$ and $\|A - B\| < \epsilon$, then $B \in \Phi(X)$ and $N(B) \subseteq B(X)$.

Assume that $\alpha(A) > 0$ and $\beta(A) > 0$. Then there are $x_0, y_0 \in X$ with $x_0 \neq 0$, $x_0 \in N(A)$, $y_0 \notin A(X)$ and $\|Ay_0\| = \frac{\epsilon}{2}$. It follows that $y_0 \notin N(A)$. The Hahn-Banach theorem shows that there is $x^* \in X^*$ such that

$$\alpha = x^*(x_0) \neq 0,\; x^*(y_0) = 0 \quad \text{and} \quad \|x^*\| = 1\,.$$

Define $B \in \mathscr{L}(X)$ by

$$Bx = Ax + x^*(x)Ay_0 \qquad (x \in X)\,.$$

Then $\|(A-B)x\| \leq \frac{\epsilon}{2}\|x\|$, thus $\|A-B\| < \epsilon$. By $(*)$, $B \in \Phi(X)$ and $N(B) \subseteq B(X)$. Since $B(X) \subseteq A(X)$, $N(B) \subseteq A(X)$. We have $y_0 - x_0/\alpha \in N(B)$, thus $y_0 \in A(X) + N(A) = A(X)$, a contradiction. □

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