ON A QUESTION OF MBEKHTA

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Abstract. The present paper deals with a question of M. Mbekhta concerning partial isometries on Banach spaces.

1. Introduction

Throughout this paper, \( X \) shall denote a Banach space and \( \mathcal{L}(X) \) the algebra of all bounded linear operators on \( X \). \( X^* \) denotes the dual space of \( X \). For an operator \( T \in \mathcal{L}(X) \) we write \( T^* \) for its adjoint, \( N(T) \) for its kernel and \( T(X) \) for its range.

We will say that \( T \in \mathcal{L}(X) \) has a generalized inverse if there is an operator \( S \in \mathcal{L}(X) \) for which

\[
TST = T \quad \text{and} \quad STS = S.
\]

The operator \( S \) is called a generalized inverse of \( T \). We recall that in general a generalized inverse is not unique and that \( T \) has a generalized inverse if and only if \( N(T) \) and \( T(X) \) are closed and complemented subspaces of \( X \) (see for instance, [3]). Observe that if (1.1) holds then \( TS, ST, I - TS \) and \( I - ST \) are projections, \( T(X) = TS(X), S(X) = ST(X), N(T) = (I - ST)(X) \) and \( N(S) = (I - TS)(X) \), hence

\[
X = T(X) \oplus N(S)
\]

and

\[
X = S(X) \oplus N(T).
\]

A bounded linear operator \( T \) on a Hilbert space is said to be a partial isometry provided that \( \|Tx\| = \|x\| \) for every \( x \in N(T)^\perp \), that is

\[
TT^*T = T.
\]

In this case \( T \) is a contraction (see Chapter 13 in [5] for details).

In [7] M. Mbekhta has given the following characterization of partial isometries on Hilbert spaces:

1.1. Theorem. If \( T \) is a contraction on a Hilbert space, then the following are equivalent:

(1) \( T \) is a partial isometry;
(2) \( T \) has a contractive generalized inverse.
Since assertion (2) of Theorem 1.1 does not depend on the structure of a Hilbert space, Theorem 1.1 suggests the following definition of a partial isometry on a Banach space. This definition is due to M. Mbekhta [7].

1.2. Definition. An operator $T \in \mathcal{L}(X)$ is called a partial isometry if $T$ is a contraction and admits a generalized inverse which is a contraction.

Remarks.

(1) As mentioned by Mbekhta in [7], one of the disadvantages of Definition 1.2 is that, in general, an isometry on $X$ (i.e. $\|Tx\| = \|x\|$ for all $x \in X$) does not need to be a partial isometry. Indeed an isometry may not have a generalized inverse.

(2) In Definition 1.2, the contractive generalized inverse is not unique, as is shown by an example in [7, page 776].

The following proposition collects some properties of partial isometries on Banach spaces. Proofs can be found in [7].

1.3. Proposition. If $T \in \mathcal{L}(X)$ is a non-zero partial isometry and $S$ is a contractive generalized inverse of $T$ then:

(1) $\|T\| = \|S\| = \|TS\| = \|ST\| = 1$;

(2) $S(X) \subseteq \{x \in X : \|Tx\| = \|x\|\}$.

If $T$ is a partial isometry on a Hilbert space $H$ and $S$ is a contractive generalized inverse of $T$, then $S = T^*$ (see [7, Corollary 3.3]). Hence $T$ has a unique contractive generalized inverse. Furthermore, by (1.2),

(1.3) $T^*(H) = S(H) = \{x \in H : \|Tx\| = \|x\|\}$.

In view of Proposition 1.3 (2) and (1.3) the following question, due to M Mbekhta [7], arises:

1.4. Question. If $T \in \mathcal{L}(X)$ is a partial isometry on a Banach space $X$ and $S$ is a contractive generalized inverse of $T$, does

(1.4) $S(X) = \{x \in X : \|Tx\| = \|x\|\}$?

The following example, provide in [7], shows that in general (1.4) does not hold.

1.5. Example. Let $X = \mathbb{C}^2$ be equipped with the norm $||(x, y)|| = |x| + |y|$, and consider the operator

$$T = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X).$$

Take

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then it is easy to see that $T^2 = T$, $\|T\| = \|S\| = 1$ and that $TST = T$ and $STS = S$. Thus $T$ is a partial isometry and $T$ and $S$ are contractive generalized inverses of $T$. For $(0, 1) \in X$ we have $T(0, 1) = (-1, 0)$, thus $\|T(0, 1)\| = ||(0, 1)|| = 1$, but $(0, 1) \notin S(X)$.

1.6. Proposition. If $T \in \mathcal{L}(X)$ is a partial isometry, then the following assertions are equivalent:

(1) There is a contractive generalized inverse $S$ of $T$ such that (1.4) holds.

(2) (1.4) holds for every contractive generalized inverse of $T$. 

Proof. We only have to show that (1) implies (2). Hence assume that $S$ and $S_0$ are contractive generalized inverses of $T$ and that (1.4) holds for $S$. It follows from Proposition 1.3 (2) that $S_0(T(X)) \subseteq S(X)$, therefore $S_0T(X) \subseteq ST(X)$. This gives $STS_0T = S_0T$, thus $ST = S_0T$, hence $S(X) \subseteq S_0(X)$, and so $S_0(X) = S(X)$. □

In this paper we show that in the case of a strictly convex Banach space, Question 1.4 has an affirmative answer. Furthermore we show that a partial isometry on a strictly convex Banach space with a strictly convex dual space has a unique contractive generalized inverse, and we give some corollaries of these results.

2. Results

We say that the Banach space $X$ is strictly convex if the assumptions

$$x, y \in X, \|x\| = \|y\| = 1 \quad \text{and} \quad x \neq y$$

imply that $\|x + y\| < 2$.

We say that the norm of $X$ is Gâteaux-differentiable if, for all $x \in X \setminus \{0\}$ and for all $h \in X$, the limit

$$\lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t}$$

exists when $t \to 0 (t \in \mathbb{R})$. The Banach space $X$ is called smooth if its norm is Gâteaux-differentiable. The duality between strict convexity and smoothness reads as follows (see [1]):

If $X^*$ is smooth, then $X$ is strictly convex; if $X^*$ is strictly convex, then $X$ is smooth. Hence, if $X$ is reflexive, then $X$ is smooth (strictly convex) if and only if $X^*$ is strictly convex (smooth).

Examples.

1. If $X = l^p$ or $X = L^p (1 < p < \infty)$, then $X$ and $X^*$ are strictly convex (see [6, §121]).

2. Let $X = \mathbb{R}^2$ equipped with the norm

$$\|(x, y)\| = \left( x^2 + y^2 / 4 \right)^{1/2} + \frac{|y|}{2},$$

then $X$ is strictly convex, but $X^*$ is not strictly convex (see [6, Aufgabe 121.2]).

3. Each Hilbert space is strictly convex ([6, §121]).

The main results of this paper read as follows:

2.1. Theorem. If $X$ is a strictly convex Banach space and $T \in \mathcal{L}(X)$ is a partial isometry with contractive generalized inverse $S$, then

$$S(X) = \{ x \in X : \|Tx\| = \|x\| \}.$$

and

$$S_0T = ST$$

for each contractive generalized inverse $S_0$ of $T$.

2.2. Theorem. If $X$ and $X^*$ are both strictly convex and if $T \in \mathcal{L}(X)$ is a partial isometry, then $T$ has a unique contractive generalized inverse.
Remark. As an immediate consequence of Theorem 2.2 we obtain [7, Corollary 3.3]: a partial isometry on a Hilbert space has a unique contractive generalized inverse.

Proof of Theorem 2.1. We have, by Proposition 1.3 (2) that \( S(X) \subseteq \{ x \in X : \|Tx\| = \|x\| \} \). Now let \( x \in X \) and \( \|Tx\| = \|x\| \). We can assume that \( 1 = \|x\| = \|Tx\| \). By (1.2) there are \( u \in S(X) \) and \( v \in N(T) \) such that \( x = u + v \). In view of Proposition 1.3 (2) we have \( \|Tu\| = \|u\| \), thus

\[
1 = \|x\| = \|Tx\| = \|Tu\| = \|u\|.
\]

We have to show that \( v = 0 \). Assume to the contrary that \( v \neq 0 \). Then \( u \neq x \). Since \( X \) is strictly convex, it follows that \( \|x + u\| < 2 \). But

\[
1 = \|Tu\| = \|T(u + \frac{1}{2}v)\| \leq \|T\| \|u + \frac{1}{2}v\| = \|u + \frac{1}{2}v\| = \frac{1}{2} \|x + u\| < 1,
\]

a contradiction. Hence we have \( v = 0 \), and so \( x = u \in S(X) \).

Now suppose that \( S_0 \) is also a contractive generalized inverse of \( T \). Then \( S_0(X) = \{ x \in X : \|Tx\| = \|x\| \} \), thus \( S(X) = S_0(X) \). It follows that \( ST(X) = S_0T(X) \).

Since \( N(ST) = N(T) = N(S_0T) \), we get \( ST = S_0T \).

□

Proof of Theorem 2.2. Let \( S \) and \( S_0 \) be contractive generalized inverses of \( T \). Theorem 2.1 shows that \( ST = S_0T \), thus

\[
T^*S^* = T^*S_0^* .
\]

Since \( X^* \) is strictly convex and \( T^* \) is a partial isometry with contractive generalized inverses \( S^* \) and \( S_0^* \), we obtain as above that

\[
S^*T^* = S_0^*T^* .
\]

From (2.1) and (2.2) we now obtain that

\[
S^* = (S^*T^*)S^* = (S_0^*T^*)S^* = S_0^*(T^*S^*) = S_0^*T^*S_0^* = S_0^* ,
\]

therefore \( S = S_0 \).

□

2.3. Corollary. If \( X^* \) is strictly convex and if \( T \in \mathcal{L}(X) \) is a partial isometry with contractive generalized inverses \( S \) and \( S_0 \), then

\[
TS = TS_0 \quad \text{and} \quad N(S) = N(S_0) .
\]

Proof. As in the proof of Theorem 2.2 we obtain \( S^*T^* = S_0^*T^* \), thus \( (TS)^* = (TS_0)^* \). Hence \( TS = TS_0 \) and \( N(S) = N(S_0) \).

□

2.4. Corollary. If \( X \) is strictly convex, \( P \in \mathcal{L}(X) \), \( P^2 = P \) and \( \|P\| = 1 \), then we have:

1. \( P(X) = \{ x \in X : \|Px\| = \|x\| \} \);
2. if \( S \in \mathcal{L}(X) \), \( PSP = P \), \( SPS = S \) and \( \|S\| = 1 \), then

\[
S^2 = S , \quad SP = P \quad \text{and} \quad PS = S .
\]

Proof. Since \( \|P\| = 1 \), \( P \) is a partial isometry on \( X \) and \( P \) is a contractive generalized inverse of itself. Thus, (1) follows from Theorem 2.1.

For the proof of (2) observe that \( S \) is a contractive generalized inverse of \( P \), therefore; by Theorem 2.1, \( SP = P^2 = P \). From this we get

\[
S^2 = SPS(PS)S = SPSPS = SPS = S .
\]
Therefore $S$ is a partial isometry with contractive generalized inverses $S$ and $P$. Theorem 2.1 shows now that $S^2 = PS$, hence $S = PS$.

2.5. Corollary. Suppose that $X$ and $X^*$ are strictly convex and that $Y \neq \{0\}$ is a closed and complemented subspace of $X$. Then there is at most one projection $P \in \mathcal{L}(X)$ such that $\|P\| = 1$ and $P(X) = Y$.

Proof. Let $P$ and $Q$ be projections with $\|P\| = \|Q\| = 1$ and $P(X) = Q(X) = Y$. Then $P = QP$ and $Q = PQ$, thus $P = P^2 = P(QP)$ and $Q = Q^2 = Q(PQ)$. This shows that $P$ is a partial isometry with contractive generalized inverses $P$ and $Q$. By Theorem 2.2 it results that $P = Q$. □

2.6. Corollary. Let $T \in \mathcal{L}(X)$ be a partial isometry.

1. If $X$ is strictly convex and $T$ right invertible, then there is exactly one right inverse of $T$ with norm 1.

2. If $X^*$ is strictly convex and $T$ is left invertible, then there is exactly one left inverse of $T$ with norm 1.

Proof. (1) Let $S$ and $S_0$ be right inverses of $T$ such that $\|S\| = \|S_0\| = 1$. Then $TS = TS_0 = I$. It follows that $S$ and $S_0$ are contractive generalized inverses of $T$. Using Theorem 2.1 we obtain $ST = S_0T$. Hence $S_0 = S_0TS_0 = STS_0 = S$.

(2) Let $S$ and $S_0$ be left inverses of $T$ with $\|S\| = \|S_0\| = 1$. Then $S^* = S_0$ are right inverses of $T^*$ with $\|S^*\| = \|S_0\| = 1$. By (1), $S^* = S_0$, therefore $S = S_0$. □

Definitions.

1. An operator $U \in \mathcal{L}(X)$ is called hermitian if $\|\exp(itU)\| = 1$ for every $t \in \mathbb{R}$.

2. Let $T \in \mathcal{L}(X)$. We will say that $T^+ \in \mathcal{L}(X)$ is the Moore-Penrose inverse of $T$ if $T^+$ is a generalized inverse of $T$ and the projections $TT^+$ and $T^+T$ are hermitian.

3. $T \in \mathcal{L}(X)$ is called an MP-partial isometry if $T$ is a contraction and admits a contractive Moore-Penrose inverse (see [7]).

Remarks.

1. A bounded linear operator has at most one Moore-Penrose inverse (see [8]).

2. It is well-known that a bounded linear operator $U$ on a Hilbert space is hermitian if and only if $U = U^*$ (see [2]).

3. If $T \in \mathcal{L}(X)$ is an MP-partial isometry, then $T$ is a partial isometry in the sense of Definition 1.2.

2.7. Corollary. Let $T \in \mathcal{L}(X)$ be an MP-partial isometry and $S$ a contractive generalized inverse of $T$.

1. If $X$ is strictly convex, then $ST = T^+T$.

2. If $X^*$ is strictly convex, then $TS = TT^+$.

3. If $X$ and $X^*$ are strictly convex, then $S = T^+$.

Proof. (1) follows from Theorem 2.1 and (2) follows from Corollary 2.3. (3) is obtained from Theorem 2.2. □

Question. (see [7, p. 780]) Let $T \in \mathcal{L}(X)$ be an MP-partial isometry. Does $T^+(X) = \{x \in X : \|Tx\| = \|x\|\}$?

The following example gives a negative answer to this question.
Example. Let $X = \mathbb{C}^2$ be equipped with the norm $\| (x, y) \| = \max \{|x|, |y|\}$ and consider the operator

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X).$$

Then $T^2 = T$ and $\|T\| = 1$, therefore $T$ is a contractive generalized inverse for itself. It is easy to see that

$$\exp(itT) = \begin{pmatrix} e^{it} & 0 \\ 0 & 1 \end{pmatrix},$$

thus $T$ is hermitian. Therefore $T$ is an MP-partial isometry and $T^+ = T$. Take $(x, y) = (1, 1)$, then $T(1, 1) = (1, 0)$ and $\|T(1, 1)\| = 1 = \|(1, 1)\|$, but $(1, 1) \notin T^+(X)$.

If $T \in \mathcal{L}(X) \setminus \{0\}$ has a generalized inverse $S$, then $S \neq 0$ and $\|T\| = \|TST\| \leq \|T\|^2 \|S\|$, thus $\|T\| \|S\| \geq 1$.

We say that $T \in \mathcal{L}(X)$ is a \textit{generalized partial isometry} if $T = 0$ or if $T$ has a generalized inverse $S$ such that $\|T\| \|S\| = 1$. Clearly, a partial isometry is a generalized partial isometry.

There are no restrictions on the norm for generalized partial isometries, every $\lambda I$ is a generalized partial isometry, where $\lambda \in \mathbb{C}$.

2.8. Corollary. Suppose that $T \in \mathcal{L}(X) \setminus \{0\}$ is a generalized partial isometry.

1. If $X$ is strictly convex and $S$ and $S_0$ are generalized inverses of $T$ such that $\|T\| \|S\| = \|T\| \|S_0\| = 1$, then

$$S(X) = \{ x \in X : \|Tx\| = \|T\| \|x\| \}$$

and

$$ST = S_0T.$$

2. If $X$ and $X^*$ are both strictly convex, then there is exactly one generalized inverse $S$ of $T$ with $\|T\| \|S\| = 1$.

Proof. Let $\alpha = \|T\|^{-1}$, $T_1 = \alpha T$, $S_1 = \frac{1}{\alpha} S$ and $S_2 = \frac{1}{\alpha} S_0$. Then $T_1 S_1 T_1 = T_1$, $S_1 T_1 S_1 = S_1$, $\|T_1\| = 1$ and $\|S_1\| = 1$ ($i = 1, 2$). Hence $T_1$ is a partial isometry with contractive generalized inverses $S_1$ and $S_2$.

(1) Since $S(X) = S_1(X)$, we derive from Theorem 2.1 that $S(X) = \{ x \in X : \|T_1 x\| = \|x\| \} = \{ x \in X : \|Tx\| = \|T\| \|x\| \}$. Furthermore we obtain $S_1 T_1 = S_2 T_1$, thus $ST = S_0 T$.

(2) In view of Theorem 2.2 we get $S_1 = S_2$, hence $S = S_0$. \qed

For an operator $T \in \mathcal{L}(X) \setminus \{0\}$ the \textit{reduced minimum modulus} is defined by

$$\gamma(T) = \inf \{ \|Tx\| : x \in X, \text{dist}(x, N(T)) = 1 \}.$$

It is a classical fact that $\gamma(T) > 0$ if and only if $T(X)$ is closed, and that $\gamma(T) = \gamma(T^*)$ (see [4] or [6]).

A proof of the following proposition can be found in [7].

2.9. Proposition. Let $T \in \mathcal{L}(X) \setminus \{0\}$ and $S \in \mathcal{L}(X)$ be a generalized inverse of $T$. Then

$$\frac{1}{\|S\|} \leq \gamma(T) \leq \frac{\|TS\| \|ST\|}{\|S\|}.$$
If $T$ is as in Proposition 2.9, then
\[
\gamma(T) \geq \sup \left\{ \frac{1}{\|S\|} : S \in \mathcal{L}(X), TST = T, STS = S \right\}.
\]

2.10. Corollary. If $T \in \mathcal{L}(X) \setminus \{0\}$ is a generalized partial isometry then
\[
\gamma(T) = \|T\|.
\]

Proof. Let $S$ be a generalized inverse of $T$ such that $\|T\| \|S\| = 1$ and $\|TS\| \leq 1$, hence, by Proposition 2.9,
\[
\|T\| = \frac{1}{\|S\|} \leq \gamma(T) \leq 1 = \frac{1}{\|S\|} = \|T\|.
\]
\[
\Box
\]

We say that $T \in \mathcal{L}(X)$ is a semi-Fredholm operator if $\dim N(T) < \infty$ or $\operatorname{codim} T(X) < \infty$.

The following result is well-known in the case of partial isometries on Hilbert spaces ([5, Problem 101]).

2.11. Theorem. Let $X$ be an arbitrary Banach space.

(1) If $T \in \mathcal{L}(X) \setminus \{0\}$ is a generalized partial isometry, $U \in \mathcal{L}(X)$ and $\dim N(T) < \dim N(U)$, then
\[
\|T - U\| \geq \|T\|.
\]

(2) If $T_1, T_2 \in \mathcal{L}(X)$ are generalized partial isometries and $\|T_1 - T_2\| < \min\{\|T_1\|, \|T_2\|\}$, then
\[
\dim N(T_1) = \dim N(T_2) \quad \text{and} \quad \operatorname{codim} T_1(X) = \operatorname{codim} T_2(X).
\]

Proof. (1) Since $T$ is semi-Fredholm and $\|T\| = \gamma(T)$, we have $\|T\| \leq \|T - U\|$ by [4, Theorem V.1.6]. (2) follows immediately from (1) by duality. \[
\Box
\]

2.12. Corollary. If the generalized partial isometry $T \in \mathcal{L}(X)$ is semi-Fredholm and $\dim N(T) \neq \operatorname{codim} T(X)$, then
\[
\|T - S\| \geq \min\{\|T\|, \|T\|^{-1}\}
\]
for each generalized inverse $S$ of $T$ with $\|T\| \|S\| = 1$.

Proof. Assume to the contrary that $\|T - S\| < \min\{\|T\|, \|T\|^{-1}\} = \min\{\|T\|, \|S\|\}$. It follows from Theorem 2.11 that
\[
\dim N(S) = \dim N(T) \quad \text{and} \quad \operatorname{codim} S(X) = \operatorname{codim} T(X).
\]
But (1.2) shows that $\dim N(S) = \operatorname{codim} T(X)$, thus $\dim N(T) = \operatorname{codim} T(X)$, a contradiction. \[
\Box
\]
References


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