DRAZIN INVERSES OF OPERATORS WITH RATIONAL RESOLVENT

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ABSTRACT. Let $A$ be a bounded linear operator on a Banach space such that the resolvent of $A$ is rational. If $0$ is in the spectrum of $A$, then it is well-known that $A$ is Drazin invertible. In this paper we investigate spectral properties of the Drazin inverse of $A$. For example we show that the Drazin inverse of $A$ is a polynomial in $A$.

1. Introduction and Terminology

In this paper $X$ is always a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on $X$. For $A \in \mathcal{L}(X)$ we write $N(A)$ for its kernel and $A(X)$ for its range. We write $\sigma(A)$, $\rho(A)$ and $R_\lambda(A)$ for the spectrum, the resolvent set and the resolvent operator $(A - \lambda)^{-1}$ of $A$, respectively. The ascent of $A$ is denoted by $\alpha(A)$ and the descent of $A$ is denoted by $\delta(A)$.

An operator $A \in \mathcal{L}(X)$ is Drazin invertible if there is $C \in \mathcal{L}(X)$ such that

(i) $CAC = C$,  
(ii) $AC = CA$

and

(iii) $A^\nu+1C = A^\nu$ for some nonnegative integer $\nu$.

In this case $C$ is uniquely determined (see [2]) and is called the Drazin inverse of $A$. The smallest nonnegative integer $\nu$ such that (iii) holds is called the index $i(A)$ of $A$. Observe that

$0 \in \rho(A) \iff A$ is Drazin invertible and $i(A) = 0$.

The following proposition tells us exactly which operators are Drazin invertible with index $> 0$:

1.1. Proposition. Let $A \in \mathcal{L}(X)$ and let $\nu$ be a positive integer. Then the following assertions are equivalent:

1. $A$ is Drazin invertible and $i(A) = \nu$.
2. $\alpha(A) = \delta(A) = \nu$.
3. $R_\lambda(A)$ has a pole of order $\nu$ at $\lambda = 0$.

Proof. [2, §5.2] and [3, Satz 101.2].

The next result we will use frequently in our investigations.

1.2. Proposition. Suppose that $A \in \mathcal{L}(X)$ is Drazin invertible, $i(A) = \nu \geq 1$, $P$ is the spectral projection of $A$ associated with the spectral set $\{0\}$ and that $C$ is the Drazin inverse of $A$. Then

1. $P = I - AC$,  
2. $N(C) = N(A^\nu) = P(X)$,  
3. $C(X) = N(P) = A^\nu(X)$,
4. $C$ is Drazin invertible, $i(C) = 1$, $ACA$ is the Drazin inverse of $C$,

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(3) $0 \in \sigma(C)$ and $\sigma(C) \setminus \{0\} = \{ \frac{1}{\lambda} : \lambda \in \sigma(A) \setminus \{0\} \}$.

Proof. We have $P = I - AC$, $N(A^\nu) = P(X)$ and $\sigma(C) \setminus \{0\} = \{ \frac{1}{\lambda} : \lambda \in \sigma(A) \setminus \{0\} \}$ by [2, § 5.2]. It is clear that $0 \in \sigma(C)$. From Proposition 1.1 and [3, Satz 101.2] we get $N(P) = A^\nu(X)$. If $x \in X$ then $Cx = 0 \Leftrightarrow Px = x$, hence $N(C) = P(X)$. From $P = I - AC = I - CA$ it is easily seen that $N(P) = C(X)$, Let $B = ACA$. Then

$$CBC = CACAC = CAC = C, \quad CB = CACA = ACAC = BC$$

and

$$BCB = ACACACA = ACACA = ACA = B.$$ 

This shows that $C$ is Drazin invertible, $B$ is the Drazin inverse of $C$ and that $i(C) = 1$. \hfill \Box

Now we introduce the class of operators which we will consider in this paper. We say that $A \in \mathcal{L}(X)$ has a rational resolvent if

$$R_\lambda(A) = \frac{P(\lambda)}{q(\lambda)}$$

where $P(\lambda)$ is a polynomial with coefficients in $\mathcal{L}(X)$, $q(\lambda)$ is polynomial with coefficients in $\mathbb{C}$ and where $P$ and $q$ have no common zeros.

We use the symbol $\mathcal{F}(X)$ to denote the subclass of $\mathcal{L}(X)$ consisting of those operators whose resolvent is rational.

For $A \in \mathcal{L}(X)$ let $\mathcal{H}(A)$ be the set of all functions $f : \triangle(f) \to \mathbb{C}$ such that $\triangle(f)$ is an open set in $\mathbb{C}$, $\sigma(A) \subseteq \triangle(f)$ and $f$ is holomorphic on $\triangle(f)$. For $f \in \mathcal{H}(A)$ the operator $f(A) \in \mathcal{L}(X)$ is defined by the usual operational calculus (see [3] or [4]).

The following proposition collects some properties of operators in $\mathcal{F}(X)$. An operator $A \in \mathcal{L}(X)$ is called algebraic if $p(A) = 0$ for some nonzero polynomial $p$.

1.3. Proposition. Let $A \in \mathcal{L}(X)$.

(1) $A \in \mathcal{F}(X)$ if and only if $\sigma(A)$ consists of a finite number of poles of $R_\lambda(A)$.

(2) $A \in \mathcal{F}(X)$ if and only if $A$ is algebraic.

(3) If $\dim A(X) < \infty$, then $A \in \mathcal{F}(X)$.

(4) If $A \in \mathcal{F}(X)$ and $f \in \mathcal{H}(A)$, then $f(A) = p(A)$ for some polynomial $p$.

(5) If $A \in \mathcal{F}(X)$, the $p(A) \in \mathcal{F}(X)$ for every polynomial $p$.

Proof. [4, Chapter V. 11] \hfill \Box

1.4. Corollary. Suppose that $A \in \mathcal{F}(X)$ and $0 \in \rho(A)$. Then $A^{-1} \in \mathcal{F}(X)$ and $A^{-1}$ is a polynomial in $A$.

Proof. Define the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $f(\lambda) = \lambda^{-1}$. Then $f \in \mathcal{H}(A)$ and $f(A) = A^{-1}$. Now apply Proposition 1.3 (4) and (5). \hfill \Box

Remark. That $A \in \mathcal{F}(X)$ and $0 \in \rho(A)$ implies $A^{-1} \in \mathcal{F}(X)$ is also shown in [1, Theorem 2]. In Section 2 of the present paper we will give a further proof of this fact.
2. Drazin inverses of operators in $\mathcal{F}(X)$

Throughout this section $A$ will be an operator in $\mathcal{F}(X)$ and $\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}$, where $\lambda_1, \ldots, \lambda_k$ are the distinct poles of $R_\lambda(A)$ of orders $m_1, \ldots, m_k$ (see Proposition 1.3 (1)).

Recall that

$$m_j = \alpha(A - \lambda_j) = \delta(A - \lambda_j) \ (j = 1, \ldots, k).$$

Let

$$m_A(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}. \tag{2.1}$$

By [4, Theorem V. 10.7],

$$m_A(A) = 0.$$  

The polynomial $m_A$ is called the minimal polynomial of $A$. It follows from [4, Theorem V. 10.7] that $m_A$ divides any other polynomial $p$ such that $p(A) = 0$. In what follows we always assume that $m_A$ has degree $n$, thus $n = m_1 + \cdots + m_k$ and that $m_A$ has the representations (2.1) and

$$m_A(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_{n-1} \lambda^{n-1} + \lambda^n. \tag{2.2}$$

Observe that

$$0 \in \rho(A) \iff a_0 \neq 0$$

and that

$$0 \text{ is a pole of order } \nu \geq 1 \text{ of } R_\lambda(A) \iff a_0 = \cdots = a_{\nu - 1} = 0 \text{ and } a_\nu \neq 0.$$  

Now we are in a position to state our first result. Recall from Proposition 1.1 that if $\lambda_0 \in \sigma(A)$, then $A - \lambda_0$ is Drazin invertible.

**2.1. Theorem.** If $\lambda_0 \in \sigma(A)$ and if $C$ is the Drazin inverse of $A - \lambda_0$, then there is a scalar polynomial $p$ such that $C = p(A)$.

**Proof.** Without loss of generality we can assume that $\lambda_0 = \lambda_1 = 0$. Let $\nu = m_1$. Then we have

$$m_A(\lambda) = a_\nu \lambda^\nu + a_{\nu+1} \lambda^{\nu+1} + \cdots + a_{n-1} \lambda^{n-1} + \lambda^n$$

and that $a_\nu \neq 0$. Let

$$q_1(\lambda) = -\frac{1}{a_\nu}(a_{\nu+1} + a_{\nu+2} \lambda + \cdots + \lambda^{n-(\nu+1)}).$$

Then

$$A^{\nu + 1}q_1(A) = -\frac{1}{a_\nu}(a_{\nu+1} A^{\nu+1} + a_{\nu+2} A^{\nu+2} + \cdots + A^{n})$$

$$= -\frac{1}{a_\nu}(m_A(A) - a_\nu A^{\nu}) = A^{\nu}.$$  

Let $B = q_1(A)$. Then $A^{\nu + 1}B = A^{\nu}$ and $BA = AB$. For the Drazin inverse $C$ we have

$$A^{\nu + 1}C = A, \ CAC = C \quad \text{and} \quad CA = AC.$$

Thus

$$A^{\nu + 1}(B - C) = A^{\nu + 1}B - A^{\nu + 1}C = A^{\nu} - A^{\nu} = 0$$

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This shows that \((B - C)(X) \subseteq N(A^{\nu+1})\). By Proposition 1.1, \(\alpha(A) = \nu\), thus \((B - C)(X) \subseteq N(A^{\nu})\), therefore \((B - C)(X) \subseteq P_{1}(X)\), where \(P_{1}\) denotes the spectral projection of \(A\) associated with the spectral set \(\{0\}\) (see Proposition 1.2). Since \(P_{1} = I - AC = I - CA\), it follows that
\[
B - C = P_{1}(B - C) = P_{1}B - P_{1}C = P_{1}B - (I - CA)C = P_{1}B - C + CAC = P_{1}B,
\]
thus
\[
C = B - P_{1}B.
\]
We have \(P_{1} = f(A)\) for some \(f \in \mathcal{H}(A)\). By Proposition 1.3 (4), \(f(A) = q_{2}(A)\) for some polynomial \(q_{2}\). Now define the polynomial \(p\) by \(P = q_{1} - q_{2}q_{1}\). It results that
\[
p(A) = q_{1}(A) - q_{2}(A)q_{1}(A) = B - P_{1}B = C.
\]
\[\square\]

2.2. Corollary. If \(\lambda_{0} \in \sigma(A)\) and if \(C\) is the Drazin inverse of \(A - \lambda_{0}\), then \(C \in \mathcal{F}(X)\).

Proof. Theorem 2.1 and Proposition 1.3 (5).

2.3. Corollary. Let \(A\) be a complex square matrix and \(\lambda_{0}\) a characteristic value of \(A\). Then the Drazin inverse of \(A - \lambda_{0}\) is a polynomial in \(A\).

Proof. Theorem 2.1 and Proposition 1.3 (3).

Let \(T \in \mathcal{L}(X)\). An operator \(S \in \mathcal{L}(X)\) is called a pseudo inverse of \(T\) provided that \(TST = T\). In general the set of all pseudo inverses of \(T\) is infinite and this set consists of all operators of the form
\[
STS + U - STUTS
\]
where \(U \in \mathcal{L}(X)\) is arbitrary (see \([2, \text{Theorem } 2.3.2]\)). Observe that if \(T\) is Drazin invertible with \(i(T) = 1\), then the Drazin inverse of \(T\) is a pseudo inverse of \(T\).

2.4. Corollary. If \(\lambda_{0} \in \sigma(A)\), then the following assertions are equivalent:

1. \(\lambda_{0}\) is a simple pole of \(R_{\lambda}(A)\);
2. there is a pseudo inverse \(B\) of \(A - \lambda_{0}\) such that
   \[
   B(A - \lambda_{0}) = (A - \lambda_{0})B;
   \]
3. there is a polynomial \(p\) such that \(p(A)\) is a pseudo inverse of \(A - \lambda_{0}\).

Proof. (1) \(\Leftrightarrow\) (2): Proposition 1.1.
(1) \(\Rightarrow\) (3): We can assume that \(\lambda_{0} = 0\). Let \(q_{1}\) and \(B\) as in the proof of Theorem 2.1. Then \(A^{2}B = A\) and \(AB = BA\), hence \(ABA = A\).
(3) \(\Rightarrow\) (1): Again we can assume that \(\lambda_{0} = 0\). With \(B = p(A)\) we have \(ABA = A\) and \(AB = BA\). Set \(C = BAB\), then
\[
ACA = A,\ CAC = C\quad\text{and}\quad AC = CA.
\]
This shows that \(C\) is the Drazin inverse of \(A\) and that \(i(A) = 1\). By Proposition 1.1, \(\lambda_{0} = 0\) is a simple pole of \(R_{\lambda}(A)\).

\[\square\]

2.5. Corollary. Let \(X\) be a complex Hilbert space and suppose that \(N \in \mathcal{L}(X)\) is normal and that \(\sigma(N)\) is finite. We have:

1. \(N \in \mathcal{F}(X)\),

\[\square\]
If $\lambda_0 \in \sigma(N)$, then there is a polynomial $p$ such that
\[(N - \lambda_0)p(N)(N - \lambda_0) = N - \lambda_0.
\]

Proof. By [3, Satz 111.2], each point in $\sigma(N)$ is a simple pole of $R_\lambda(N)$, thus $N \in \mathcal{F}(X)$. Now apply Corollary 2.4.

Our results suggest the following

**Question.** If $A \in \mathcal{F}(X)$ and if $B$ is a pseudo inverse such that $AB = BA$, does there exist a polynomial $p$ with $B = p(A)$?

The answer is negative:

**Example.** Consider the square matrix
\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}.
\]

It is easy to see that the minimal polynomial of $A$ is given by $m_A(\lambda) = \lambda^2 - 3\lambda = \lambda(\lambda - 3)$, hence $\sigma(A) = \{0, 3\}$ and $A^2 = 3A$. Let
\[
B = \frac{1}{3} \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

Then $AB = BA = \frac{1}{3}A$, thus $ABA = \frac{1}{3}A^2 = A$, hence $B$ is a pseudo inverse of $A$. Since $A^2 = 3A$, any polynomial in $A$ has the form $\alpha I + \beta A$ with $\alpha, \beta \in \mathbb{C}$. But there are no $\alpha$ and $\beta$ such that $B = \alpha I + \beta A$. An easy computation shows that the Drazin inverse of $A$ is given by $\frac{1}{3}A$ and that $i(A) = 1$.

If $0$ is a simple pole of $R_\lambda(A)$, then we have seen in Corollary 2.4 that $A$ has a pseudo inverse. If $0$ is a pole of $R_\lambda(A)$ of order $\geq 2$, then, in general $A$ does not have a pseudo inverse, as the following example shows.

**Example.** Let $T \in \mathcal{L}(X)$ be any operator with $T(X)$ not closed (of course $X$ must be infinite dimensional). Define the operator $A \in \mathcal{L}(X \oplus X)$ by the matrix
\[
A = \begin{pmatrix}
0 & 0 \\
T & 0
\end{pmatrix}.
\]

Then the range of $A$ is not closed. By [2, Theorem 2.1], $A$ has no pseudo inverse. From $A^2 = 0$ it follows that $A \in \mathcal{F}(X \oplus X)$ and that $0$ is a pole of order 2 of $R_\lambda(A)$.

Now we return to the investigations of our operator $A \in \mathcal{F}(X)$. To this end we need the following propositions.

**2.6. Proposition.** Suppose that $T \in \mathcal{L}(X)$, $0 \in \rho(T)$, $\lambda \in \mathbb{C} \setminus \{0\}$ and that $k$ is a nonnegative integer. Then:

1. $N(T - \lambda)^k = N((T^{-1} - \frac{1}{\lambda})^k)$;
2. $\alpha(T - \lambda) = \alpha(T^{-1} - \frac{1}{\lambda})$.

Proof. We only have to show that $N((T - \lambda)^k \subseteq N((T^{-1} - \frac{1}{\lambda})^k)$. Take $x \in N((T - \lambda)^k)$. Then $0 = (T - \lambda)^k x$, thus $0 = (T^{-1})^k(T - \lambda)^k x = (1 - \lambda T^{-1})^k x$, hence $x \in N((T^{-1} - \frac{1}{\lambda})^k)$. 

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2.7. Proposition. Suppose that $T \in \mathcal{L}(X)$, $0 \in \sigma(T)$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $k$ is a nonnegative integer. Furthermore suppose that $T$ is Drazin invertible and that $C$ is the Drazin inverse of $T$. Then:

(1) $N((T - \lambda)^k) = N((C - \frac{1}{\lambda})^k)$;

(2) $\alpha(T - \lambda) = \alpha(C - \frac{1}{\lambda})$.

Proof. (2) follows from (1).

2.8. Proposition. Let $\nu = i(T)$. We use induction. First we show that $N(T - \lambda) = N(C - \frac{1}{\lambda})$.

Let $x \in N(T - \lambda)$, then $Tx = \lambda x$ and $T^\nu x = \lambda^\nu x$. We have

$$\lambda^2 C^2 x = C^2 Tx = CTCx = Cx,$$

hence $C(1 - \lambda C)x = 0$, thus $(1 - \lambda C)x \subseteq N(C)$. By Proposition 1.2, $N(C) = N(T^\nu)$, therefore

$$0 = T^\nu (1 - \lambda C)x = (1 - \lambda C)T^\nu x = (1 - \lambda C)\lambda^\nu x,$$

therefore $x \in N(C - \frac{1}{\lambda})$. Now let $x \in N(C - \frac{1}{\lambda})$. From $Cx = \frac{1}{\lambda}x$ we see that $x \in C(X) = N(P)$, where $P$ is as in Proposition 1.2. From $P = I - TC$ we get $x = TCx = T(\frac{1}{\lambda}x)$, thus $Tx = \lambda x$, hence $x \in N(T - \lambda)$. Now suppose that $n$ is a positive integer and that $N((T - \lambda)^n) = N((C - \frac{1}{\lambda})^n)$.

Take $x \in N((T - \lambda)^n+1)$. Then $(T - \lambda)x \in N((T - \lambda)^n) = N((C - \frac{1}{\lambda})^n)$, thus

$$0 = (C - \frac{1}{\lambda})^n(T - \lambda)x = (T - \lambda)(C - \frac{1}{\lambda})^n x.$$

This gives

$$(C - \frac{1}{\lambda})^n x \in N(T - \lambda) = N(C - \frac{1}{\lambda}),$$

therefore $x \in N((C - \frac{1}{\lambda})^{n+1})$. Similar arguments show that $N((C - \frac{1}{\lambda})^{n+1}) \subseteq N((T - \lambda)^{n+1})$. \qed

In what follows we use the notation of the beginning of this section.

Recall that we have $\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}$. If $0 \in \sigma(A)$, then we always assume that $\lambda_1 = 0$, hence $\sigma(A) \setminus \{0\} = \{\lambda_2, \ldots, \lambda_k\}$.

2.9. Theorem. Suppose that $0 \in \rho(A)$.

(1) If the minimal polynomial $m_A$ has the representation (2.1), then the minimal polynomial $m_{A^{-1}}$ of $A^{-1}$ is given by

$$m_{A^{-1}}(\lambda) = (\lambda - \frac{1}{\lambda_1})^{m_1} \cdots (\lambda - \frac{1}{\lambda_k})^{m_k}.$$

(2) If the minimal polynomial $m_A$ has the representation (2.2), then $m_{A^{-1}}$ is given by

$$m_{A^{-1}}(\lambda) = \frac{1}{a_0} + \frac{a_{n-1}}{a_0} \lambda + \cdots + \frac{a_1}{a_0} \lambda^{n-1} + \lambda^n.$$
Then multiplying (2.3) by $C_q$ now define the polynomial $q$ by

$$q(\lambda) = \frac{1}{a_0} + \frac{a_{n-1}}{a_0} \lambda + \cdots + \frac{a_1}{a_0} \lambda^{n-1} + \lambda^n.$$  

Then

$$a_0 A^n q(A^{-1}) = A^n (a_0 (A^{-1})^n + a_1 (A^{-1})^{n-1} + \cdots + a_{n-1} A^{-1} + 1) = m_A(A) = 0.$$  

Since $a_0 \neq 0$ and $0 \in \rho(A)$, it results that $q(A^{-1}) = 0$. Because of degree of $q = n = \deg m_A$, we get $q = m_A$. \hfill \Box

**Remark.** The proof just given shows that there is a polynomial $q$ such that $q(A^{-1}) = 0$. Therefore we have a simple proof for the fact that $A^{-1} \in \mathcal{F}(X)$.

**2.10. Theorem.** Suppose that $0 \in \sigma(A)$ and that 0 is a pole of $R_\lambda(A)$ of order $\nu \geq 1$. Let $C$ denote the Drazin inverse of $A$ (recall from Corollary 2.2 that $C \in \mathcal{F}(X)$).

1. If $m_A$ has the representation (2.1), then

$$m_C(\lambda) = \lambda (\lambda - \frac{1}{x_2})^{m_2} \cdots (\lambda - \frac{1}{x_k})^{m_k}.$$  

2. If $m_A$ has the representation (2.2), then

$$m_C(\lambda) = \frac{1}{a_\nu} \lambda + \frac{a_{\nu-1}}{a_\nu} \lambda^2 + \cdots + \frac{a_{\nu+1}}{a_\nu} \lambda^{n+1-(\nu+1)} + \lambda^{n+1-\nu}.$$  

**Proof.** Proposition 2.7 gives

$$\alpha(C - \frac{1}{x_j}) = \alpha(A - \lambda_j) = m_j \quad (j = 2, \ldots, k).$$

By Proposition 1.1 and Proposition 1.2, $\alpha(C) = 1$. Thus (1) is valid.

We have

$$m_A(\lambda) = a_\nu \lambda^\nu + a_{\nu+1} \lambda^{\nu+1} + \cdots + a_{n-1} \lambda^{n-1} + \lambda^n,$$

hence

$$0 = m_A(A) = a_\nu A^\nu + a_{\nu+1} A^{\nu+1} + \cdots + a_{n-1} A^{n-1} + A^n.$$  

If $\nu \leq l \leq n$, then

$$C^{n+1} A^l = C^{n+1-l} C^l A^l = C^{n+1-l} (CA)^l = C^{n+1-l}CA = C^{n+1-l}CAC = C^{n+1-l}.$$  

Then multiplying (2.3) by $C^{n+1}$, it follows that

$$0 = a_\nu C^{n+1-\nu} + a_{\nu+1} C^{n+1-(\nu+1)} + \cdots + a_{n-1} C^2 + C.$$  

Now define the polynomial $q$ by

$$q(\lambda) = \frac{1}{a_\nu} \lambda + \frac{a_{\nu-1}}{a_\nu} \lambda^2 + \cdots + \frac{a_{\nu+1}}{a_\nu} \lambda^{n+1-(\nu+1)} + \lambda^{n+1-\nu}.$$  

Then $q(C) = 0$. Since degree of $q = n + 1 - \nu = 1 + m_2 + \cdots + m_k = \deg m_C$, we get $q = m_C$. \hfill \Box

**2.11. Corollary.** With the notation in Theorem 2.10 we have

$$C(A - \lambda_2)^{m_2} \cdots (A - \lambda_k)^{m_k} = 0.$$  

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Proof. Let \( D = (A - \lambda_2)^{m_2} \cdots (A - \lambda_k)^{m_k} \). From \( A^\nu D = m_A(A) = 0 \) we see that \( D(X) \subseteq N(A^\nu) \).
Since \( N(A^\nu) = N(C) \) (Proposition 1.2), \( CD = 0 \).

\[ \square \]

Notation. \( X^* \) denotes the dual space of \( X \) and we write \( T^* \) for the adjoint of an operator \( T \in \mathcal{L}(X) \). Recall from [4, Theorem IV. 8.4] that

\[ (2.4) \quad T(X) = N(T^*)^\perp \quad (T \in \mathcal{L}(X)). \]

2.12. Proposition. Suppose that \( T \in \mathcal{L}(X), \lambda \in \mathbb{C} \setminus \{0\} \) and that \( j \) is a nonnegative integer.

1. If \( 0 \in \rho(T) \), then

\[ (T - \lambda)^j(X) = (T^{-1} - \frac{1}{\lambda})^j(X). \]

2. If \( 0 \in \sigma(T) \), if \( T \) is Drazin invertible and if \( C \) denotes the Drazin inverse of \( T \), then

\[ (T - \lambda)^j(X) = (C - \frac{1}{\lambda})^j(X). \]

Proof. (1) Let \( y = (T - \lambda)^j x \in (T - \lambda)^j(X) \) \( (x \in X) \). Then

\[ (T^{-1} - \frac{1}{\lambda})^j T^j x = ((T^{-1} - \frac{1}{\lambda})T)^j x = (1 - \frac{1}{\lambda})^j x \]

\[ = \frac{(-1)^j}{\lambda^j} (T - \lambda)^j x = \frac{(-1)^j}{\lambda^j} y, \]

therefore \( y \in (T^{-1} - \frac{1}{\lambda})^j(X) \).

(2) Let \( \nu = i(T) \). Then \( T^{\nu+1} C = T^\nu, \) \( TC = CT \) and \( CTC = C \). Hence

\[ (T^*)^{\nu+1} C^* = (T^*)^\nu, \quad T^*C^* = C^*T^* \quad \text{and} \quad C^*T^*C^* = C^*. \]

Thus \( T^* \) is Drazin invertible and \( C^* \) is the Drazin inverse of \( T^* \). By Proposition 2.7,

\[ N((T^* - \lambda)^j) = N((C^* - \frac{1}{\lambda})^j), \]

therefore the result follows in view of (2.4).

\[ \square \]

2.13. Corollary.

1. If \( 0 \in \rho(A) \), then

\[ (A - \lambda_j)^{m_j}(X) = (A^{-1} - \frac{1}{\lambda_j})^{m_j}(X) \quad (j = 1, \ldots, k). \]

2. If \( 0 \in \sigma(A) \) is a pole of order \( \nu \geq 1 \) of \( R_A(A) \) and if \( C \) is the Drazin inverse of \( A \), then

\[ A^\nu(X) = C(X) \]

and

\[ (A - \lambda_j)^{m_j}(X) = (C - \frac{1}{\lambda_j})^{m_j}(X) \quad (j = 2, \ldots, k). \]

Proof. (1) is a consequence of Proposition 2.12.
(2) That \( A^\nu(X) = C(X) \) is a consequence of Proposition 1.2. Now let \( j \in \{2, \ldots, k\} \). Because of Proposition 1.1 and Theorem 2.10 we see that

\[ \alpha(C - \frac{1}{\lambda_j}) = \delta(C - \frac{1}{\lambda_j}) = m_j = \alpha(A - \lambda_j) = \delta(A - \lambda_j). \]

By [3, Satz 101.2], the subspaces

\[ (A - \lambda_j)^{m_j}(X) \quad \text{and} \quad (C - \frac{1}{\lambda_j})^{m_j}(X) \]

are closed. Now apply Proposition 2.12.

\[ \square \]
For \( j = 1, \ldots, k \) let \( P_j \) denote the spectral projection of \( A \) associated with the spectral set \( \{ \lambda_j \} \). Observe that

\[
P_i P_j = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad P_1 + \cdots + P_k = 1.
\]

If \( 0 \in \rho(A) \), then denote by \( Q_j \) the spectral projection of \( A^{-1} \) associated with the spectral set \( \{ \frac{1}{\lambda_j} \} \) \( (j = 1, \ldots, k) \). If \( 0 \in \sigma(A) \) and if \( C \) is the Drazin inverse, then denote by \( Q_1 \) the spectral projection of \( C \) associated with the spectral set \( \{ 0 \} \) and by \( Q_j \) the spectral projection of \( C \) associated with the spectral set \( \{ \frac{1}{\lambda_j} \} \) \( (j = 2, \ldots, k) \).

2.14. Corollary. \( P_j = Q_j \) \( (j = 1, \ldots, k) \).

Proof. By [3, Satz 101.2], we have

\[
P_j(X) = N((A - \lambda_j)^{m_j}) \quad \text{and} \quad N(P_j) = (A - \lambda_j)^{m_j}(X)
\]

\( (j = 1, \ldots, k) \). If \( 0 \in \rho(A) \), then

\[
Q_j(X) = N((A^{-1} - \frac{1}{\lambda_j})^{m_j}) \quad \text{and} \quad N(Q_j) = (A^{-1} - \frac{1}{\lambda_j})^{m_j}(X)
\]

\( (j = 1, \ldots, k) \). Now apply Proposition 2.6 and Corollary 2.13 (1) to get

\[
P_j(X) = Q_j(X) \quad \text{and} \quad N(P_j) = N(Q_j),
\]

hence \( P_j = Q_j \) \( (j = 1, \ldots, k) \).

Now let \( 0 \in \sigma(A) \). By Proposition 1.2, Proposition 2.7, Corollary 2.13 (2) and [3, Satz 101.2], we derive

\[
P_1(X) = N(C) = Q_1(X), \quad N(P_1) = C(X) = N(Q_1),
\]

\[
P_j(X) = N((C - \frac{1}{\lambda_j})^{m_j}) = Q_j(X)
\]

and

\[
N(P_j) = (C - \frac{1}{\lambda_j})^{m_j}(X) = N(Q_j)
\]

\( (j = 2, \ldots, k) \). Hence \( P_j = Q_j \) \( (j = 1, \ldots, k) \). \( \square \)

For \( A \) we have the representation

\[
A = \sum_{j=1}^{k} \lambda_j P_j + N,
\]

where \( N \in \mathcal{L}(X) \) is nilpotent and \( N = \sum_{j=1}^{k} (A - \lambda_j)P_j \) (see [4, Chapter V. 11]). If \( p = \max\{m_1, \ldots, m_k\} \), then it is easily seen that \( N^p = 0 \). If \( A \) has only simple poles, then \( N = 0 \).

2.15. Corollary. \( 1 \) If \( 0 \in \rho(A) \), then there is a nilpotent operator \( N_1 \in \mathcal{L}(X) \) with

\[
A^{-1} = \sum_{j=1}^{k} \frac{1}{\lambda_j} P_j + N_1
\]

\( 2 \) If \( 0 \in \sigma(A) \) and if \( C \) is the Drazin inverse of \( A \), then

\[
C = \sum_{j=2}^{k} \frac{1}{\lambda_j} P_j + N_1,
\]

where \( N_1 \in \mathcal{L}(X) \) is nilpotent.

Proof. Corollary 2.14. \( \square \)
With the notation of Corollary 2.15 (2) we have
\[ AC = 1 - P_1, \quad CP_1 = 0 \]
(see Proposition 1.2) and
\[ ACA = (1 - P_1) \left( \sum_{j=2}^{k} \lambda_j P_j + N \right) = A - P_1 \left( \sum_{j=2}^{k} \lambda_j P_j + N \right) = A - P_1 N, \]

hence
\[ A = ACA + P_1 N, \quad P_1 N \text{ is nilpotent} \]
and
\[ (ACA)P_1 N = ACP_1 AN = 0 = NACP_1 A = P_1 N(ACA). \]

Recall that \( ACA \) is the Drazin inverse of \( C \) and that \( i(ACA) = 1. \)

The following more general result holds:

**2.16. Theorem.** Suppose that \( T \in \mathcal{L}(X) \) is Drazin invertible, \( i(T) = \nu \geq 1 \) and that \( C \) is the Drazin inverse of \( T \). Then there is a nilpotent \( N \in \mathcal{L}(X) \) such that
\[ T = TCT + N, \quad N(TCT) = (TCT)N = 0 \quad \text{and} \quad N^\nu = 0. \]

This decomposition is unique in the following sense: if \( S, N_1 \in \mathcal{L}(X) \), \( S \) is Drazin invertible, \( i(S) = 1, \) \( N_1 \) is nilpotent, \( N_1 S = SN_1 = 0 \) and if \( T = S + N_1 \), then \( S = TCT \) and \( N = N_1. \)

**Proof.** Let \( N = T - TCT \), then \( N^\nu = (T(1-CT))^\nu = T^\nu(1-CT)^\nu = T^\nu(1-CT) = T^\nu - T^\nu C = T^\nu - T^\nu + T^\nu = 0. \)

For the uniqueness of the decomposition we only have to show that \( S = TCT \). There is \( R \in \mathcal{L}(X) \) such that
\[ SRS = S, \quad RSR = R \quad \text{and} \quad SR = RS. \]

Consequently,
\[ N_1 R = N_1 RSR = N_1 SR^2 = 0 = R^S SN_1 = RN_1, \]

hence
\[ TR = (S + N_1) R = SR = RS = R(S + N_1) = RT. \]

Now let \( n \) be a nonnegative integer such that \( N_1^n = 0. \) It follows that, since \( SN_1 = 0 = N_1 S, \)
\[ T^n = (S + N_1)^n = S^n + N_1^n = S^n. \]

We can assume that \( n \geq \nu. \) Thus
\[ T^{n+1} R = S^{n+1} R = S^{n-1} SRS = S^n = T^n. \]

Furthermore we have \( TR = RT \) and
\[ RTR = R(S + N_1) R = RSR = R, \]

hence \( R = C. \) With \( S_1 = TCT \) we get
\[ S_1 RS_1 = TCT TCTCT = TCT = S_1, \]
\[ RS_1 R = C TCT = C TCT = RTR = R \]

and
\[ S_1 R = TCTC = CTCT = RS_1. \]

This shows that \( S = S_1 = TCT. \)
References


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