

## NULL CONTROLLABILITY FOR PARABOLIC EQUATIONS WITH DYNAMIC BOUNDARY CONDITIONS

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**ABSTRACT.** We prove null controllability for linear and semilinear heat equations with dynamic boundary conditions of surface diffusion type. The results are based on a new Carleman estimate for this type of boundary conditions.

**1. Introduction.** In this paper we establish the null controllability of linear and semilinear parabolic equations with dynamic boundary conditions of surface diffusion type. The prototype of such problems is

$$\begin{aligned} \partial_t y - \Delta y &= \mathbb{1}_\omega v(t, x) && \text{in } (0, T) \times \Omega, \\ \partial_t y_\Gamma - \Delta_\Gamma y_\Gamma + \partial_\nu y &= 0 && \text{on } (0, T) \times \Gamma, \\ y_\Gamma(t, x) &= y|_\Gamma(t, x) && \text{on } (0, T) \times \Gamma, \\ (y, y_\Gamma)|_{t=0} &= (y_0, y_{0,\Gamma}) && \text{in } \Omega \times \Gamma. \end{aligned} \tag{1.1}$$

For all given  $T > 0$ ,  $\omega \Subset \Omega$  and initial data  $y_0 \in L^2(\Omega)$  in the bulk and  $y_{0,\Gamma} \in L^2(\Gamma)$  on the boundary, we want to find a control  $v \in L^2((0, T) \times \omega)$  such that the solution satisfies

$$y(T, \cdot) = 0 \quad \text{in } \overline{\Omega}.$$

Here  $\Omega \subset \mathbb{R}^N$  is a bounded domain with smooth boundary  $\Gamma = \partial\Omega$ ,  $N \geq 2$ , and the control region  $\omega$  is an *arbitrary* nonempty open subset which is strictly contained in  $\Omega$  (i.e.,  $\overline{\omega} \subset \Omega$ ). Further,  $y|_\Gamma$  denotes the trace of a function  $y : \Omega \rightarrow \mathbb{R}$ ,  $\nu$  is the outer unit normal field,  $\partial_\nu y := (\nu \cdot \nabla y)|_\Gamma$  is the normal derivative at  $\Gamma$ , and  $\Delta_\Gamma$  designates the Laplace-Beltrami operator on  $\Gamma$ . In our main results we study more general problems involving time-dependent potentials, forcing and semilinear terms in the interior and at the boundary, as well as boundary control, see Theorems 4.2, 4.4 and 4.5.

The term  $\partial_t y_\Gamma - \Delta_\Gamma y_\Gamma$  models the tangential diffusive flux on the boundary which is coupled to the diffusion equation in the bulk by the normal derivative  $\partial_\nu y$ . In (1.1) we treat this problem as a coupled system of dynamic equations for  $y$  and  $y_\Gamma$ , with side condition  $y|_\Gamma = y_\Gamma$ . Sometimes this type of boundary conditions is called of Wentzell type. Dynamic surface and interface processes have attracted a lot of attention in recent years in the mathematical and applied literature, see [3, 5, 7, 8, 14, 15, 16, 17, 18, 26, 30, 34]. In particular, for the mathematical theory of surface diffusion boundary conditions we refer to [5, 8, 14, 26, 30, 34]. In Section 2 we discuss the  $L^2$ -based solution theory for (1.1) and its generalizations as needed in the context of null controllability. Here we look at existence, uniqueness and regularity of strong, mild and distributional solutions. Since we deal with time-dependent potential terms in  $L^\infty$ , we include proofs.

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2010 *Mathematics Subject Classification.* Primary: 93B05. Secondary: 35K20, 93B07.

*Key words and phrases.* Parabolic problems, dynamic boundary conditions, surface diffusion, Carleman estimate, null controllability, observability estimate.

We thank the Deutsche Forschungsgemeinschaft which supported this research within the grants ME 3848/1-1 and SCHN 570/4-1. M.M. thanks L.M. for a very pleasant stay in Marrakesh, where parts of this work originated.

We state our main result ensuring the null controllability of (1.1), see Theorem 4.2. We emphasize that the initial data  $y_0$  and  $y_{0,\Gamma}$  on  $\Omega$  and  $\Gamma$  need not be related.

**Theorem 1.1.** *For each  $T > 0$ , each nonempty open set  $\omega \Subset \Omega$  and all initial data  $y_0 \in L^2(\Omega)$  and  $y_{0,\Gamma} \in L^2(\Gamma)$  there is a control  $v \in L^2((0, T) \times \omega)$  such that the unique mild solution  $y$  of (1.1) satisfies  $y(T, \cdot) = 0$  on  $\bar{\Omega}$ .*

Null controllability results of this type are known for Dirichlet and for inhomogeneous or nonlinear Neumann boundary conditions (also called Robin or Fourier boundary conditions), see e.g. [4, 6, 9, 10, 13, 19, 24] and the survey article [11]. Our Theorem 4.2 actually treats a more general version of the control problem (1.1) with nonautonomous potential terms and inhomogeneities in the bulk and on the boundary. We also show null controllability for a semilinear variant of (1.1) including globally Lipschitz nonlinearities both in the bulk and on the boundary, see Theorem 4.4. Observe that one could drop the assumption that  $\bar{\omega} \subset \Omega$  by simply extending the control  $v$  by 0. We will not do this since one would lose information in this way. As in the case of Dirichlet or Neumann boundary conditions, in Theorem 4.5 we obtain null controllability with controls acting on a part  $\Gamma_0$  of the boundary by means of an auxiliary domain control problem on an enlarged spatial domain, see [4, 6, 13, 19]. However, in our case one needs more regularity for the solution of the problem to implement this strategy. We partly solve this problem by means of local  $L^p$ -regularity, but we do not obtain the anticipated regularity of the solution and the boundary control  $v$  near the boundary of the support of  $v$ . In view of the results in [6, 9, 10, 11], we expect that one can extend the results to nonlinearities with slightly superlinear growth and containing gradient terms, as in [12], using more involved regularity theory for (1.1). In this work, we have considered the heat equation with constant diffusion coefficients, but as in the case of static boundary conditions presumably our results also hold for general elliptic second order operators, with diffusion coefficients even depending on time. Recently, many results are obtained even in the case where diffusion coefficients degenerate at the boundary, see [1] and the references therein.

We are not aware of results on null controllability for parabolic problems with dynamical boundary conditions. Optimal control problem and approximate controllability for such equations were treated in [2] and [18] in the case of global controls; i.e.,  $\omega = \Omega$ . In [22], approximate boundary controllability of a one-dimensional heat equation with dynamical boundary conditions was studied by completely different methods.

Theorem 1.1 relies on a Carleman estimate for the inhomogeneous dual problem corresponding to (1.1), which is proved in Lemma 3.2. Roughly speaking, this estimate bounds a weighted  $L^2$ -norm of the solution  $\varphi$  to the dual problem by a weighted  $L^2$ -norm of the inhomogeneities and of the restriction  $\varphi|_\omega$ . These weights tend to zero exponentially as  $t \rightarrow 0$  and  $t \rightarrow T$ . The proof of our Carleman estimate follows the known strategy of the Dirichlet case, see [11], but the dynamic boundary condition leads to various new boundary terms. Some of these enter in the final estimate, a few cancel, and others can be controlled using the smoothing effect of the surface diffusion in (1.1). Since we have to use the smoothing effect, we cannot treat the corresponding equations without surface diffusion, see Remark 3.3 for more details.

With the Carleman estimate at hand, standard arguments (see Proposition 4.1) yield the observability estimate

$$\|\varphi(0, \cdot)\|_{L^2(\Omega)}^2 + \|\varphi_\Gamma(0, \cdot)\|_{L^2(\Gamma)}^2 \leq C \int_0^T \int_\omega |\varphi|^2 dx dt$$

for the solution  $\varphi$  of the dual homogeneous backward problem

$$\begin{aligned} -\partial_t \varphi - d\Delta \varphi + a(t, x)\varphi &= 0 && \text{in } \Omega_T, \\ -\partial_t \varphi_\Gamma - \delta \Delta_\Gamma \varphi_\Gamma + d\partial_\nu \varphi + b(t, x)\varphi_\Gamma &= 0 && \text{on } \Gamma_T, \\ \varphi_\Gamma(t, x) &= \varphi(t, x) && \text{on } (0, T) \times \Gamma, \\ (\varphi(T, \cdot), \varphi_\Gamma(T, \cdot)) &= (\varphi_T, \varphi_{T,\Gamma}) && \text{in } \Omega \times \Gamma. \end{aligned}$$

One calls this property the final state observability since the observation on  $\omega$  controls the state at the final time. By duality, the observability estimate for  $\varphi$  then yields the null controllability

of (1.1) as stated in Theorem 1.1, see Theorem 4.2. We refer to [33] for a discussion of various controllability and observability concepts.

The proof of the null controllability of the semilinear equation relies on Schauder's fixed point theorem. To set up this fixed point argument, we construct a control with minimal weighted  $L^2$ -norm for the inhomogenous linear system involving nonautonomous potential terms. This optimization problem is solved in Proposition 4.3. In its proof we adopt the methods of Imanuvilov's seminal paper [19] to the case of dynamic boundary conditions.

This paper is organized as follows. In Section 2 we introduce the functional analytic setting and state basic results for (versions of) (1.1) concerning well-posedness in the framework of strong, mild and distributional solutions. The Carleman estimate is proved in Section 3 and the null controllability results are obtained in Section 4.

**Acknowledgment.** We thank the referee for a valuable comments and for pointing out a mistake.

**2. The initial-boundary value problem.** Let  $T > 0$  and a bounded domain  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 2$ , with smooth boundary  $\Gamma = \partial\Omega$  and outer unit normal field  $\nu$  on  $\Gamma$  be given. We write

$$\Omega_T = (0, T) \times \Omega, \quad \omega_T = (0, T) \times \omega, \quad \text{and} \quad \Gamma_T = (0, T) \times \Gamma,$$

where  $\omega \Subset \Omega$  is open. In this section we present wellposedness and regularity properties of solutions of the inhomogeneous linear system

$$\partial_t y - d\Delta y + a(t, x)y = f(t, x) \quad \text{in } \Omega_T, \quad (2.1)$$

$$\partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d\partial_\nu y + b(t, x)y_\Gamma = g(t, x) \quad \text{on } \Gamma_T, \quad (2.2)$$

$$y_\Gamma(t, x) = y|_\Gamma(t, x) \quad \text{on } \Gamma_T, \quad (2.3)$$

$$(y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}) \quad \text{in } \Omega \times \Gamma, \quad (2.4)$$

for given coefficients  $d, \delta > 0$ ,  $a \in L^\infty(\Omega_T)$  and  $b \in L^\infty(\Gamma_T)$ . We include proofs since the potentials depend on time and are just  $L^\infty$  and since we also deal with very weak solution concepts.

**2.1. Function spaces.** The Lebesgue measure on  $\Omega$  and the surface measure on  $\Gamma$  are denoted by  $dx$  and  $dS$ , respectively. We consider the real Hilbert spaces (and tacitly their complexifications if necessary)

$$\mathbb{L}^2 := L^2(\Omega) \times L^2(\Gamma), \quad \mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) : y|_\Gamma = y_\Gamma\} \quad \text{for } k \in \mathbb{N},$$

$$\mathbb{E}_1(t_0, t_1) := H^1(t_0, t_1; \mathbb{L}^2) \cap L^2(t_0, t_1; \mathbb{H}^2) \quad \text{for } t_1 > t_0 \text{ in } \mathbb{R}, \quad \mathbb{E}_1 := \mathbb{E}_1(0, T).$$

The scalar product on  $\mathbb{L}^2$  is given by

$$\langle (y, y_\Gamma), (z, z_\Gamma) \rangle_{\mathbb{L}^2} = \langle y, z \rangle_{L^2(\Omega)} + \langle y_\Gamma, z_\Gamma \rangle_{L^2(\Gamma)}.$$

Further,  $H^k(\Omega)$  are the usual  $L^2$ -based Sobolev spaces over  $\Omega$ . The spaces  $H^k(\Gamma)$  are defined via local coordinates, see e.g. Definition 3.6.1 in [32]. At a few points we will also need the fractional order spaces  $H^s(\Omega)$  and  $H^s(\Gamma)$  with noninteger  $s \geq 0$ . For our purposes it suffices to define them as interpolation spaces

$$H^s(\Omega) = (L^2(\Omega), H^2(\Omega))_{s/2, 2}, \quad H^s(\Gamma) = (L^2(\Gamma), H^2(\Gamma))_{s/2, 2}, \quad s \in [0, 2],$$

where  $(\cdot, \cdot)_{s/2, 2}$  denotes the real interpolation functor, see Chapter 1 and Theorem 4.3.1/2 of [32] or Chapter 1 of [25]. As a consequence, we obtain the interpolation inequalities

$$\|y\|_{H^s(\Omega)} \leq C \|y\|_{L^2(\Omega)}^{1-s/2} \|y\|_{H^2(\Omega)}^{s/2}, \quad \|y_\Gamma\|_{H^s(\Gamma)} \leq C \|y_\Gamma\|_{L^2(\Gamma)}^{1-s/2} \|y_\Gamma\|_{H^2(\Gamma)}^{s/2}, \quad s \in [0, 2], \quad (2.5)$$

see Theorem 1.3.3 of [32].

For every  $s > \frac{1}{2}$ , the trace operator on  $\Gamma$  is continuous and surjective from  $H^s(\Omega)$  to  $H^{s-1/2}(\Gamma)$  and has a continuous right-inverse  $\mathcal{E}_\Gamma : H^{s-1/2}(\Gamma) \rightarrow H^s(\Omega)$ , see e.g. Theorem 4.7.1 in [32]. Given  $s_0 > \frac{1}{2}$ , the right-inverse can be chosen to be independent of  $s < s_0$ . The normal derivative  $\partial_\nu y = (\nu \cdot \nabla y)|_\Gamma$  is thus continuous from  $H^s(\Omega)$  to  $H^{s-3/2}(\Gamma)$  for each  $s > \frac{3}{2}$ .

Finally, for open sets  $\omega \subset \Omega$ , we consider  $L^2(\omega)$  as a closed subspace of  $L^2(\Omega)$  by extending functions on  $\omega$  by zero to  $\Omega$ .

**2.2. The Laplace-Beltrami operator.** We refer to Chapter 3 of [20] or Sections 2.4 and 5.1 of [31] for more details and proofs. The operator  $\Delta_\Gamma$  on  $\Gamma$  is given by

$$(\Delta_\Gamma y) \circ g = \frac{1}{\sqrt{\det G}} \sum_{i,j=1}^{N-1} \partial_i (\sqrt{\det G} g^{ij} \partial_j (y \circ g))$$

in local coordinates  $g$ , where  $G = (g_{ij})$  is the metric tensor corresponding to  $g$  and  $G^{-1} = (g^{ij})$  denotes its inverse. However, in this paper we will not use this local formula, but rather the surface divergence theorem

$$\int_\Gamma \Delta_\Gamma y z \, dS = - \int_\Gamma \langle \nabla_\Gamma y, \nabla_\Gamma z \rangle_\Gamma \, dS, \quad y \in H^2(\Gamma), \, z \in H^1(\Gamma), \quad (2.6)$$

where  $\nabla_\Gamma$  is the surface gradient and  $\langle \cdot, \cdot \rangle_\Gamma$  is the Riemannian inner product of tangential vectors on  $\Gamma$ . The Laplace-Beltrami operator with domain  $H^2(\Gamma)$  is self-adjoint and negative on  $L^2(\Gamma)$ , cf. p. 309 of [31], and it thus generates an analytic  $C_0$ -semigroup on  $L^2(\Gamma)$ . Hence,  $\|y\|_{L^2(\Gamma)} + \|\Delta_\Gamma y\|_{L^2(\Gamma)}$  defines an equivalent norm on  $H^2(\Gamma)$ . Moreover,  $\|y\|_{L^2(\Gamma)} + \|\nabla_\Gamma y\|_{L^2(\Gamma)}$  gives an equivalent norm on  $H^1(\Gamma)$ .

**2.3. The Laplacian with surface diffusion boundary conditions.** On  $\mathbb{L}^2$  we consider the linear operator

$$A = \begin{pmatrix} d\Delta & 0 \\ -d\partial_\nu & \delta\Delta_\Gamma \end{pmatrix}, \quad D(A) = \mathbb{H}^2.$$

Observe that  $\mathbb{E}_1 = H^1(0, T; \mathbb{L}^2) \cap L^2(0, T; D(A))$ . Strong solutions of (2.1)–(2.4) will belong to  $\mathbb{E}_1$ . We show that  $A$  is selfadjoint and negative. The well-posedness and regularity results for the underlying evolution equations rely on this fact, which mainly follows from a result in [27].

**Proposition 2.1.** *The operator  $A$  is densely defined, self-adjoint and generates an analytic  $C_0$ -semigroup  $(e^{tA})_{t \geq 0}$  on  $\mathbb{L}^2$ . We further have  $(\mathbb{L}^2, \mathbb{H}^2)_{1/2,2} = \mathbb{H}^1$ .*

*Proof.* It is straightforward to check that  $\{(y, y|_\Gamma) : y \in C^\infty(\bar{\Omega})\} \subset \mathbb{H}^2$  is dense in  $\mathbb{L}^2$ , hence  $A$  is densely defined. To prove the asserted properties of  $A$ , on  $\mathbb{L}^2$  we introduce the densely defined, closed, symmetric, positive sesquilinear form

$$\mathfrak{a}[(y, y_\Gamma), (z, z_\Gamma)] = \int_\Omega d\nabla y \cdot \nabla \bar{z} \, dx + \int_\Omega y \cdot \bar{z} \, dx + \int_\Gamma \delta \langle \nabla_\Gamma y_\Gamma, \nabla_\Gamma \bar{z}_\Gamma \rangle_\Gamma \, dS + \int_\Gamma \langle y_\Gamma, \bar{z}_\Gamma \rangle_\Gamma \, dS$$

on  $D(\mathfrak{a}) = \mathbb{H}^1$ . It induces a positive self-adjoint sectorial operator  $\tilde{A}$  on  $\mathbb{L}^2$  which is given as follows, see Theorems VI.2.1 and VI.2.6 in [21]. A function  $(y, y_\Gamma) \in \mathbb{H}^1$  belongs to  $D(\tilde{A})$  if and only if there is  $(f, f_\Gamma) \in \mathbb{L}^2$  such that  $\mathfrak{a}[(y, y_\Gamma), (\varphi, \varphi_\Gamma)] = \langle (f, f_\Gamma), \varphi \rangle_{\mathbb{L}^2}$  for all  $(\varphi, \varphi_\Gamma) \in \mathbb{H}^1$ , and in this case  $\tilde{A}(y, y_\Gamma) = (f, f_\Gamma)$ . Integration by parts and (2.6) yield that  $\mathbb{H}^2 \subset D(\tilde{A})$  and  $\tilde{A}(y, y_\Gamma) = (Id - A)(y, y_\Gamma)$  for all  $(y, y_\Gamma) \in \mathbb{H}^2$ ; i.e.,  $\tilde{A}$  is an extension of  $Id - A$ .

By Lemma A.1 of Miranville-Zelik [27],  $\lambda - A$  is surjective for all  $\lambda > 0$ . Hence,  $Id - A = \tilde{A}$  and so  $A$  is self-adjoint and generates an analytic  $C_0$ -semigroup. Theorem VI.2.23 of [21] then says that  $D(A^{1/2}) = D(\mathfrak{a}) = \mathbb{H}^1$ . Moreover, Theorem 4.36 of [25] yields  $D(A^{1/2}) = (\mathbb{L}^2, D(A))_{1/2,2}$ , so that  $\mathbb{H}^1 = (\mathbb{L}^2, \mathbb{H}^2)_{1/2,2}$ .  $\square$

We note a few basic properties of the space  $\mathbb{E}_1$ .

**Proposition 2.2.** *The following assertions are true.*

(a) *For each  $\tau \in [0, T]$ , the trace space of  $\mathbb{E}_1$  at  $t = \tau$  equals  $\mathbb{H}^1$ . We further have*

$$\mathbb{E}_1 \hookrightarrow C([0, T]; \mathbb{H}^1).$$

(b) *The space  $\mathbb{E}_1$  embeds compactly into  $L^2(0, T; \mathbb{L}^2)$ .*

*Proof.* (a) Corollary 1.14 of [25] implies that the trace space of  $\mathbb{E}_1$  equals  $(\mathbb{L}^2, \mathbb{H}^2)_{1/2,2}$  and that  $\mathbb{E}_1 \hookrightarrow C([0, T]; (\mathbb{L}^2, \mathbb{H}^2)_{1/2,2})$ . Thus the first assertion is a consequence of Proposition 2.1.

(b) Since  $\mathbb{H}^2 \hookrightarrow H^2(\Omega) \times H^2(\Gamma)$  is compact in  $\mathbb{L}^2$ , the compactness of  $\mathbb{E}_1 \hookrightarrow L^2(0, T; \mathbb{L}^2)$  follows from the Aubin-Lions lemma, see e.g. Corollary 4 in [29].  $\square$

**2.4. Existence, uniqueness and regularity of solutions.** We are concerned with the following classes of solutions of (2.1)–(2.4).

**Definition 2.3.** Let  $f \in L^2(\Omega_T)$ ,  $g \in L^2(\Gamma_T)$  and  $Y_0 := (y_0, y_{0,\Gamma}) \in \mathbb{L}^2$ .

- (a) A strong solution of (2.1)–(2.4) is a function  $Y := (y, y_\Gamma) \in \mathbb{E}_1$  fulfilling (2.1)–(2.4) in  $L^2(0, T; \mathbb{L}^2)$ .  
 (b) A mild solution of (2.1)–(2.4) is a function  $Y := (y, y_\Gamma) \in C([0, T]; \mathbb{L}^2)$  satisfying

$$Y(t) = e^{tA}Y_0 + \int_0^t e^{(t-\tau)A}[f(\tau) - a(\tau)y(\tau), g(\tau) - b(\tau)y_\Gamma(\tau)] d\tau \quad \text{for } t \in [0, T]. \quad (2.7)$$

- (c) A distributional solution of (2.1)–(2.4) is a function  $Y := (y, y_\Gamma) \in L^2(0, T; \mathbb{L}^2)$  such that for all  $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$  with  $\varphi(T, \cdot) = \varphi_\Gamma(T, \cdot) = 0$  we have

$$\begin{aligned} \int_{\Omega_T} y(-\partial_t \varphi - d\Delta \varphi + a\varphi) dx dt + \int_{\Gamma_T} y_\Gamma(-\partial_t \varphi_\Gamma - \delta \Delta_\Gamma \varphi_\Gamma + d\partial_\nu \varphi + b\varphi_\Gamma) dS dt \\ = \int_{\Omega_T} f\varphi dx dt + \int_{\Gamma_T} g\varphi_\Gamma dS dt + \int_{\Omega} y_0 \varphi(0, \cdot) dx + \int_{\Gamma} y_{0,\Gamma} \varphi_\Gamma(0, \cdot) dS. \end{aligned} \quad (2.8)$$

- (d) We call  $Y := (y, y_\Gamma) \in L^2(0, T; \mathbb{L}^2)$  a distributional solution of (2.1)–(2.4) with vanishing final value if  $Y$  satisfies (2.8) for all  $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$ .

We show below that a strong solution is a mild one and that mild and distributional solutions coincide in our setting. Since our controllability results rely on an observability estimate for a dual problem, we also have to look at the adjoint backward evolution equation

$$-\partial_t \varphi - d\Delta \varphi + a(t, x)\varphi = f(t, x) \quad \text{in } \Omega_T, \quad (2.9)$$

$$-\partial_t \varphi_\Gamma - \delta \Delta_\Gamma \varphi_\Gamma + d\partial_\nu \varphi + b(t, x)\varphi_\Gamma = g(t, x) \quad \text{on } \Gamma_T, \quad (2.10)$$

$$\varphi_\Gamma(t, x) = \varphi|_{\Gamma}(t, x) \quad \text{on } \Gamma_T, \quad (2.11)$$

$$(\varphi(T, \cdot), \varphi_\Gamma(T, \cdot)) = (\varphi_T, \varphi_{T,\Gamma}) \quad \text{in } \Omega \times \Gamma, \quad (2.12)$$

for given  $(\varphi_T, \varphi_{T,\Gamma})$  in  $\mathbb{H}^1$  or in  $\mathbb{L}^2$ ,  $f \in L^2(\Omega_T)$  and  $g \in L^2(\Gamma_T)$ . As in Definition 2.3, a *strong solution* of (2.9)–(2.12) is a function  $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$  fulfilling (2.9)–(2.12) in  $L^2(0, T; \mathbb{L}^2)$ , and a *mild solution* of (2.9)–(2.12) is a function  $(\varphi, \varphi_\Gamma) \in C([0, T]; \mathbb{L}^2)$  satisfying

$$(\varphi(t), \varphi_\Gamma(t)) = e^{(T-t)A}(\varphi_T, \varphi_{T,\Gamma}) + \int_t^T e^{(\tau-t)A}[f(\tau) - a(\tau)\varphi(\tau), g(\tau) - \beta(\tau)\varphi_\Gamma(\tau)] d\tau \quad \text{for } t \in [0, T]. \quad (2.13)$$

Considering (2.1)–(2.4) with the data  $\hat{a}(t, x) = a(T-t, x)$ ,  $\hat{b}(t, x) = b(T-t, x)$ ,  $\hat{f}(t, x) = f(T-t, x)$  and  $\hat{g}(t, x) = g(T-t, x)$ , one can pass from statements about (2.1)–(2.4) to those about (2.9)–(2.12), and vice versa, by means of the transformation  $t' = T - t$ . Hence, the following results on strong and mild solutions have straightforward analogues for the adjoint problem which can easily be proved by this transformation. We omit the details, but establish in Proposition 2.5(f) a ‘solution formula’ for homogenous backward system which is crucial for our main Theorem 4.2.

We start with strong solutions of (2.1)–(2.4). Proposition 2.2 implies that such a solution can only exist if  $y_0 \in \mathbb{H}^1$ ; i.e., the initial data on  $\Omega$  and  $\Gamma$  are related by the trace.

**Proposition 2.4.** Let  $f \in L^2(\Omega_T)$ ,  $g \in L^2(\Gamma_T)$  and  $Y_0 := (y_0, y_{0,\Gamma}) \in \mathbb{H}^1$ . Then there exists a unique strong solution  $Y := (y, y_\Gamma) \in \mathbb{E}_1$  of (2.1)–(2.4), which is also a mild solution. Given  $R > 0$ , there is a constant  $C = C(R) > 0$  such that for all  $a$  and  $b$  with  $\|a\|_\infty, \|b\|_\infty \leq R$  and all data we have

$$\|Y\|_{\mathbb{E}_1} \leq C (\|Y_0\|_{\mathbb{H}^1} + \|f\|_{L^2(\Omega_T)} + \|g\|_{L^2(\Gamma_T)}). \quad (2.14)$$

*Proof.* We set  $F = (f, g)$  and  $B(t)(\psi, \psi_\Gamma) = (a(t, \cdot)\psi, b(t, \cdot)\psi_\Gamma)$ . Since  $A$  is selfadjoint and negative and  $B(\cdot)$  is uniformly bounded, Theorem 3.1 of [28] yields the asserted unique solution  $Y := (y, y_\Gamma) \in \mathbb{E}_1$  of (2.1)–(2.4) and the estimate (2.14). Using that  $Y \in \mathbb{E}_1$  solves (2.1)–(2.4), we compute

$$Y(t) - e^{tA}Y_0 = \int_0^t \partial_s [e^{(t-s)A}Y(s)] ds = \int_0^t e^{(t-s)A}[-B(s)Y(s) + F(s)] ds \quad (2.15)$$

so that  $Y$  is also a mild solution of (2.1)–(2.4). To obtain the asserted uniformity of the constant in (2.14), we note that Theorem 3.1 of [28] gives bounded linear operators  $S(t, s)$  on  $\mathbb{L}^2$  depending strongly continuously on  $0 \leq s \leq t \leq T$  such that

$$Y(t) = S(t, 0)Y_0 + \int_0^t S(t, s)F(s) ds, \quad t \in [0, T]. \quad (2.16)$$

Taking  $F = 0$  and varying the initial time, (2.15) yields

$$S(t, \tau)Y_0 = e^{(t-\tau)A}Y_0 - \int_\tau^t e^{(t-s)A}B(s)S(s, \tau)Y_0 ds \quad (2.17)$$

at first for  $Y_0 \in \mathbb{H}^1$  and then for  $Y_0 \in \mathbb{L}^2$  by approximation. From Gronwall's inequality we now deduce that  $\|S(t, s)\| \leq C = C(R)$  for all  $0 \leq s \leq t \leq T$ . Due to (2.16), the strong solution of (2.1)–(2.4) thus satisfies

$$\|Y(t)\|_{\mathbb{L}^2} \leq C(\|Y_0\|_{\mathbb{L}^2} + \|f\|_{L^2(\Omega_T)} + \|g\|_{L^2(\Gamma_T)}) \quad (2.18)$$

for all  $t \in [0, T]$ . We further write  $\tilde{F} = BY + F$  so that  $\partial_t Y = AY + \tilde{F}$ . The estimates (2.14) for  $a = b = 0$  and (2.18) finally yield  $C = C(R)$  with

$$\begin{aligned} \|Y\|_{\mathbb{E}_1} &\leq C(\|Y_0\|_{\mathbb{H}^1} + \|\tilde{F}\|_{L^2(0, T; \mathbb{L}^2)}) \\ &\leq C(\|Y_0\|_{\mathbb{H}^1} + \|f\|_{L^2(\Omega_T)} + \|g\|_{L^2(\Gamma_T)} + R\|Y\|_{L^2(0, T; \mathbb{L}^2)}) \\ &\leq C(\|Y_0\|_{\mathbb{H}^1} + \|f\|_{L^2(\Omega_T)} + \|g\|_{L^2(\Gamma_T)}). \end{aligned} \quad \square$$

We next consider mild and distributional solutions for initial data in  $\mathbb{L}^2$ .

**Proposition 2.5.** *Let  $f \in L^2(\Omega_T)$ ,  $g \in L^2(\Gamma_T)$  and  $Y_0 := (y_0, y_{0, \Gamma}), (\varphi_T, \varphi_{T, \Gamma}) \in \mathbb{L}^2$ . Then the following assertions are true.*

- (a) *There is a unique mild solution  $Y \in C([0, T]; \mathbb{L}^2)$  of (2.1)–(2.4). The solution map  $(Y_0, f, g) \mapsto Y$  is linear and continuous from  $\mathbb{L}^2 \times L^2(\Omega_T) \times L^2(\Gamma_T)$  to  $C([0, T]; \mathbb{L}^2)$ . Moreover,  $Y$  belongs to  $\mathbb{E}_1(\tau, T)$  and solves (2.1)–(2.4) strongly on  $(\tau, T)$  with initial data  $Y(\tau)$ , for each  $\tau \in (0, T)$ . There are bounded linear operators  $S(t, s)$  on  $\mathbb{L}^2$  depending strongly continuously on  $0 \leq s \leq t \leq T$  such that*

$$Y(t) = S(t, 0)Y_0 + \int_0^t S(t, s)(f(s), g(s)) ds, \quad t \in [0, T]. \quad (2.19)$$

- (b) *Given  $R > 0$ , there is a constant  $C = C(R) > 0$  such that for all  $a$  and  $b$  with  $\|a\|_\infty, \|b\|_\infty \leq R$  and all data the mild solution of  $y$  of (2.1)–(2.4) satisfies*

$$\|Y\|_{C([0, T]; \mathbb{L}^2)} \leq C(\|Y_0\|_{\mathbb{L}^2} + \|f\|_{L^2(\Omega_T)} + \|g\|_{L^2(\Gamma_T)}). \quad (2.20)$$

- (c) *If  $Y_0 \in \mathbb{H}^1$ , then the mild solution from (a) is the strong one.*  
(d) *A function  $Y$  is a distributional solution of (2.1)–(2.4) if and only if it is a mild solution.*  
(e) *A distributional solution of  $Y$  (2.1)–(2.4) with vanishing end value satisfies  $Y(T, \cdot) = 0$ .*  
(f) *The backward problem (2.9)–(2.12) with  $f = g = 0$  has the unique mild solution given by  $(\varphi(t), \varphi_\Gamma(t)) = S(T, t)^*(\varphi_T, \varphi_{T, \Gamma})$ , where  $S(T, t)^*$  denotes the adjoint of  $S(T, t)$ . It is the unique strong solution if  $(\varphi, \varphi_\Gamma) \in \mathbb{H}^1$ .*

*Proof.* (a)–(c). A straightforward Gronwall estimate shows uniqueness of mild solutions. Theorem 3.1 of [28] gives the desired operators  $S(t, s)$  satisfying (2.19). Above we have seen that these operators also fulfill the integral equation (2.17). Inserting (2.17) into (2.19) and interchanging integrals, we see that the function given by (2.19) is a mild solution. The linearity and continuity of the solution map follows from (2.19) and the asserted regularity of  $Y$  is shown in Theorem 3.1 of [28]. Part (b) can be checked as in (2.18) above, and (c) is a consequence of Proposition 2.4 and the uniqueness in assertion (a).

(d) Let  $F \in L^2(0, T; \mathbb{L}^2)$  and  $Y_0 \in \mathbb{H}^1$ . The function  $v$  defined by  $v(t) = e^{tA}Y_0 + \int_0^t e^{(t-s)A}F(s) ds$  belongs to  $\mathbb{E}_1$  and satisfies  $\partial_t v = Av + F$  on  $(0, T)$ , due to Proposition 2.4

with  $a = b = 0$ . Using the self-adjointness of  $A$  and integration by parts in time, we thus obtain

$$\begin{aligned} \int_0^T \left\langle e^{tA}Y_0 + \int_0^t e^{(t-s)A}F(s) ds, -(\partial_t\varphi(t), \partial_t\varphi_\Gamma(t)) - A(\varphi(t), \varphi_\Gamma(t)) \right\rangle_{\mathbb{L}^2} dt \\ = \int_0^T \langle F(t), \varphi(t) \rangle_{\mathbb{L}^2} dt + \langle Y_0, (\varphi(0), \varphi_\Gamma(0)) \rangle_{\mathbb{L}^2} \end{aligned} \quad (2.21)$$

for all  $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$  with  $\varphi(T) = \varphi_\Gamma(T) = 0$ . By approximation, this identity also holds for  $Y_0 \in \mathbb{L}^2$ . Now, let  $Y \in C([0, T]; \mathbb{L}^2)$  be the mild solution of (2.1)–(2.4). With the notation of the proof of Proposition 2.4 we have

$$Y(t) = e^{tA}Y_0 + \int_0^t e^{(t-s)A}[(f(s), g(s)) - B(s)Y(s)] ds,$$

and thus (2.21) with  $F = (f, g)$  implies that

$$\begin{aligned} \int_0^T \langle Y(t), -(\partial_t\varphi(t), \partial_t\varphi_\Gamma(t)) - A(\varphi(t), \varphi_\Gamma(t)) \rangle_{\mathbb{L}^2} dt \\ + \int_0^T \left\langle \int_0^t e^{(t-s)A}B(s)Y(s) ds, -(\partial_t\varphi(t), \partial_t\varphi_\Gamma(t)) - A(\varphi(t), \varphi_\Gamma(t)) \right\rangle_{\mathbb{L}^2} dt \\ = \int_0^T \langle F(t), (\varphi(t), \varphi_\Gamma(t)) \rangle_{\mathbb{L}^2} dt + \langle Y_0, (\varphi(0), \varphi_\Gamma(0)) \rangle_{\mathbb{L}^2} \end{aligned}$$

for each  $(\varphi, \varphi_\Gamma)$  as above. Another application of (2.21) with  $Y_0 = 0$  and  $F = BY$  yields that here the second term on the left-hand side equals  $\int_0^T \langle BY(t), (\varphi(t), \varphi_\Gamma(t)) \rangle_{\mathbb{L}^2} dt = \int_0^T \langle Y(t), B(\varphi(t), \varphi_\Gamma(t)) \rangle_{\mathbb{L}^2} dt$ . Hence, the mild solution  $Y$  is a distributional solution.

To prove that a distributional solution is already the mild one, we show uniqueness of distributional solutions. Let  $Y, Z \in L^2(0, T; \mathbb{L}^2)$  be such solutions. We then obtain

$$\langle Y - Z, -(\partial_t\varphi(t), \partial_t\varphi_\Gamma(t)) - A(\varphi(t), \varphi_\Gamma(t)) + B(\varphi(t), \varphi_\Gamma(t)) \rangle_{L^2(0, T; \mathbb{L}^2)} = 0$$

for all  $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$  with  $\varphi(T) = \varphi_\Gamma(T) = 0$ . The version of Proposition 2.4 for the backward problem (2.9)–(2.12) says that for every  $(\psi, \psi_\Gamma) \in L^2(0, T; \mathbb{L}^2)$  there is a function  $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$  with  $\varphi(T) = \varphi_\Gamma(T) = 0$  such that  $(\psi, \psi_\Gamma) = -(\partial_t\varphi, \partial_t\varphi_\Gamma) - A(\varphi, \varphi_\Gamma) + B(\varphi, \varphi_\Gamma)$ . Therefore  $Y = Z$ , and uniqueness follows.

(e) Let  $Y$  be a distributional solution with vanishing end value. Parts (a) and (d) yield  $Y \in \mathbb{E}_1(\tau, T)$  and  $\partial_t Y = AY - BY + (f, g)$  on  $(\tau, T)$ , for all  $\tau \in (0, T)$ . Let  $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$  be supported in  $(0, T]$ . Integrating and using the self-adjointness of  $A$  and  $B(t)$ , we then derive

$$\begin{aligned} \langle Y(T), (\varphi(T), \varphi_\Gamma(T)) \rangle_{\mathbb{L}^2} &= \int_0^T (\langle \partial_t Y(t), (\varphi(t), \varphi_\Gamma(t)) \rangle_{\mathbb{L}^2} + \langle Y(t), (\partial_t\varphi(t), \partial_t\varphi_\Gamma(t)) \rangle_{\mathbb{L}^2}) dt \\ &= \int_0^T \langle Y(t), (\partial_t\varphi(t), \partial_t\varphi_\Gamma(t)) + A(\varphi(t), \varphi_\Gamma(t)) - B(t)(\varphi(t), \varphi_\Gamma(t)) \rangle_{\mathbb{L}^2} dt \\ &\quad + \int_0^T \langle (f(t), g(t)), (\varphi(t), \varphi_\Gamma(t)) \rangle_{\mathbb{L}^2} dt \\ &= 0 \end{aligned}$$

from (2.8). Since every  $(\psi, \psi_\Gamma) \in \mathbb{H}^2$  can be represented as  $(\psi, \psi_\Gamma) = (\varphi(T), \varphi_\Gamma(T))$  with a function  $(\varphi, \varphi_\Gamma)$  as above and  $\mathbb{H}^2$  is dense in  $\mathbb{L}^2$ , we conclude that  $Y(T) = 0$ .

(f) As explained before Proposition 2.4, we can derive from our previous results that the backward problem (2.9)–(2.12) with  $f = g = 0$  and  $(\varphi_T, \varphi_{T,\Gamma}) \in \mathbb{L}^2$  has a unique mild solution  $(\varphi, \varphi_\Gamma)$  which is the unique strong solution if  $(\varphi_T, \varphi_{T,\Gamma}) \in \mathbb{H}^1$ . Equation (3.4) in the proof of Theorem 3.1 in [28] further gives the formula

$$S(T, t)Y_0 = e^{(T-t)A}Y_0 - \int_t^T S(T, \tau)B(\tau)e^{(\tau-t)A}Y_0 d\tau$$

in  $\mathbb{L}^2$  for  $t \in [0, T]$ . Taking adjoints and using the self-adjointness of  $A$  and  $B(\tau)$ , we derive

$$S(T, t)^*(\varphi_T, \varphi_{T, \Gamma}) = e^{(T-t)A}(\varphi_T, \varphi_{T, \Gamma}) - \int_t^T e^{(\tau-t)A} B(\tau) S(T, \tau)^*(\varphi_T, \varphi_{T, \Gamma}) d\tau$$

for  $(\varphi_T, \varphi_{T, \Gamma}) \in \mathbb{L}^2$ . In view of (2.13), the function  $S(T, \cdot)^*(\varphi_T, \varphi_{T, \Gamma})$  is the mild solution of (2.9)–(2.12) with  $f = g = 0$ .  $\square$

We note that for strong solutions of (1.1) with  $v = 0$  one has the dissipation equality

$$\|Y(T)\|_{\mathbb{L}^2}^2 = \|Y_0\|_{\mathbb{L}^2}^2 - 2\|\nabla y\|_{L^2(\Omega_T)}^2 - 2\|\nabla_\Gamma y_\Gamma\|_{L^2(\Gamma_T)}^2,$$

and not only the estimate (2.20).

**3. The Carleman estimate.** In this section we prove a Carleman estimate for the backward adjoint linear problem (2.9)–(2.12), which is the key to null controllability in the linear and semi-linear case. The weights appearing in the Carleman estimate are the same as in [11] for the case of Dirichlet boundary conditions and in [13] for mixed boundary conditions. They are based on the following auxiliary function  $\eta^0$ , see Lemma 1.2 in [11] and Lemma 1.1 in [13].

**Lemma 3.1.** *Given a nonempty open set  $\omega' \Subset \Omega$ , there is a function  $\eta^0 \in C^2(\overline{\Omega})$  such that*

$$\eta^0 > 0 \quad \text{in } \Omega, \quad \eta^0 = 0 \quad \text{on } \Gamma, \quad |\nabla \eta^0| > 0 \quad \text{in } \overline{\Omega} \setminus \omega'.$$

Since  $|\nabla \eta^0|^2 = |\nabla_\Gamma \eta^0|^2 + |\partial_\nu \eta^0|^2$  on  $\Gamma$ , the function  $\eta^0$  in the lemma satisfies

$$\nabla_\Gamma \eta^0 = 0, \quad |\nabla \eta^0| = |\partial_\nu \eta^0|, \quad \partial_\nu \eta^0 \leq -c < 0 \quad \text{on } \Gamma, \quad (3.1)$$

for some constant  $c > 0$ .

Given  $\omega' \Subset \Omega$ , we take  $\lambda, m > 1$  and  $\eta^0$  with respect to  $\omega'$  as in Lemma 3.1. Following [11], we then define the weight functions  $\alpha$  and  $\xi$  by

$$\begin{aligned} \alpha(x, t) &= (t(T-t))^{-1} (e^{2\lambda m \|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}), \\ \xi(x, t) &= (t(T-t))^{-1} e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}, \end{aligned}$$

for  $x \in \overline{\Omega}$  and  $t \in (0, T)$ . Note that  $\alpha$  and  $\xi$  are  $C^2$  and strictly positive on  $(0, T) \times \overline{\Omega}$  and blow up as  $t \rightarrow 0$  and as  $t \rightarrow T$ . Moreover, the weights are constant on the boundary  $\Gamma$  so that

$$\nabla_\Gamma \alpha = 0 \quad \text{and} \quad \nabla_\Gamma \xi = 0 \quad \text{on } \Gamma. \quad (3.2)$$

We state the Carleman estimate. In the proof we follow the strategy of Lemma 1.3 of [11]. In our setting several new boundary terms arise from the dynamic boundary condition. To collect and treat them, we have to repeat some steps from [11] in modified form.

**Lemma 3.2.** *Let  $T > 0$ ,  $\omega \Subset \Omega$  be nonempty and open,  $d, \delta > 0$ ,  $a \in L^\infty(Q_T)$  and  $b \in L^\infty(\Gamma_T)$ . Choose a nonempty open set  $\omega' \Subset \omega$ . Define  $\eta^0$ ,  $\alpha$  and  $\xi$  as above with respect to  $\omega'$ . Then there are constants  $C > 0$  and  $\lambda_1, s_1 \geq 1$  such that*

$$\begin{aligned} & s^{-1} \int_{\Omega_T} e^{-2s\alpha} \xi^{-1} (|\partial_t \varphi|^2 + |\Delta \varphi|^2) dx dt + s^{-1} \int_{\Gamma_T} e^{-2s\alpha} \xi^{-1} (|\partial_t \varphi_\Gamma|^2 + |\Delta_\Gamma \varphi_\Gamma|^2) dS dt \\ & \quad + s\lambda^2 \int_{\Omega_T} e^{-2s\alpha} \xi |\nabla \varphi|^2 dx dt + s\lambda \int_{\Gamma_T} e^{-2s\alpha} \xi |\nabla_\Gamma \varphi_\Gamma|^2 dS dt \\ & \quad + s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 |\varphi|^2 dx dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |\varphi_\Gamma|^2 dS dt \\ & \quad + s\lambda \int_{\Gamma_T} e^{-2s\alpha} \xi |\partial_\nu \varphi|^2 dS dt \\ & \leq C s^3 \lambda^4 \int_{\omega_T} e^{-2s\alpha} \xi^3 |\varphi|^2 dx dt + C \int_{\Omega_T} e^{-2s\alpha} |\partial_t \varphi + d\Delta \varphi - a\varphi|^2 dx dt \\ & \quad + C \int_{\Gamma_T} e^{-2s\alpha} |\partial_t \varphi_\Gamma + \delta \Delta_\Gamma \varphi_\Gamma - d\partial_\nu \varphi - b\varphi_\Gamma|^2 dS dt \end{aligned}$$

for all  $\lambda \geq \lambda_1$ ,  $s \geq s_1$  and  $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$ . Given  $R > 0$ , the constant  $C = C(R)$  can be chosen independently of all  $a, b$  with  $\|a\|_\infty, \|b\|_\infty \leq R$ .

*Proof.* Rescaling in time, we restrict to the case  $d = 1$ . It can be seen by convolution with mollifiers in space and time that  $C^\infty([0, T] \times \overline{\Omega})$  is dense in  $\mathbb{E}_1$ . Since all terms in the asserted inequality are continuous with respect to the  $\mathbb{E}_1$ -norm, it suffices to consider smooth functions  $\varphi \in C^\infty([0, T] \times \overline{\Omega})$ . For such functions we write  $\varphi$  instead of  $\varphi_\Gamma$ . Throughout  $C$  denotes a generic constant which does not depend on  $\lambda, s, a, b$ , and  $\varphi$  subject to the assumptions of the lemma.

*Step 1. Change of variables.* Let  $\varphi \in C^\infty([0, T] \times \overline{\Omega})$ ,  $\lambda \geq \lambda_1 \geq 1$  and  $s \geq s_1 \geq 1$  be given. Define

$$\psi := e^{-s\alpha}\varphi, \quad f = e^{-s\alpha}(\partial_t\varphi + \Delta\varphi - a\varphi), \quad g = e^{-s\alpha}(\partial_t\varphi + \delta\Delta_\Gamma\varphi - \partial_\nu\varphi - b\varphi).$$

Observe that these functions vanish exponentially at  $t = 0$  and  $t = T$ . We determine the problem solved by  $\psi$ . We first expand the spatial derivatives of  $\alpha$  by the chain rule to bring  $\eta^0$  into play, but we do not expand  $\partial_t\alpha$ . We calculate

$$\nabla\alpha = -\nabla\xi = -\lambda\xi\nabla\eta^0, \quad (3.3)$$

$$\Delta\alpha = -\lambda^2\xi|\nabla\eta^0|^2 - \lambda\xi\Delta\eta^0,$$

$$\partial_t\psi = e^{-s\alpha}\partial_t\varphi - s\psi\partial_t\alpha,$$

$$\nabla\psi = e^{-s\alpha}\nabla\varphi - s\psi\nabla\alpha = e^{-s\alpha}\nabla\varphi + s\lambda\psi\xi\nabla\eta^0, \quad (3.4)$$

$$\begin{aligned} \Delta\psi &= e^{-s\alpha}\Delta\varphi + \nabla(e^{-s\alpha}) \cdot \nabla\varphi - s\psi\Delta\alpha - s\nabla\psi \cdot \nabla\alpha \\ &= e^{-s\alpha}\Delta\varphi - 2s\nabla\psi \cdot \nabla\alpha - s^2\psi|\nabla\alpha|^2 - s\psi\Delta\alpha. \end{aligned}$$

On  $\Omega_T$  this yields transformed evolution equations

$$\begin{aligned} \partial_t\psi + \Delta\psi - a\psi &= f - s\psi(\partial_t\alpha + \Delta\alpha) - 2s\nabla\psi \cdot \nabla\alpha - s^2\psi|\nabla\alpha|^2 \\ &= f - s\psi\partial_t\alpha + s\lambda^2\psi\xi|\nabla\eta^0|^2 + s\lambda\psi\xi\Delta\eta^0 + 2s\lambda\xi\nabla\psi \cdot \nabla\eta^0 - s^2\lambda^2\psi\xi^2|\nabla\eta^0|^2. \end{aligned} \quad (3.5)$$

Similarly, using (3.1) and (3.2), on  $\Gamma_T$  we obtain

$$\partial_t\psi + \delta\Delta_\Gamma\psi - \partial_\nu\psi - b\psi = g - s\psi\partial_t\alpha - s\lambda\psi\xi\partial_\nu\eta^0. \quad (3.6)$$

Extending the corresponding decomposition in [11], we rewrite the equations (3.5) and (3.6) as

$$M_1\psi + M_2\psi = \tilde{f} \quad \text{in } \Omega_T, \quad N_1\psi + N_2\psi = \tilde{g} \quad \text{on } \Gamma_T, \quad (3.7)$$

with the abbreviations

$$M_1\psi = -2s\lambda^2\psi\xi|\nabla\eta^0|^2 - 2s\lambda\xi\nabla\psi \cdot \nabla\eta^0 + \partial_t\psi = (M_1\psi)_1 + (M_1\psi)_2 + (M_1\psi)_3,$$

$$M_2\psi = s^2\lambda^2\psi\xi^2|\nabla\eta^0|^2 + \Delta\psi + s\psi\partial_t\alpha = (M_2\psi)_1 + (M_2\psi)_2 + (M_2\psi)_3,$$

$$N_1\psi = \partial_t\psi + s\lambda\psi\xi\partial_\nu\eta^0 = (N_1\psi)_1 + (N_1\psi)_2,$$

$$N_2\psi = \delta\Delta_\Gamma\psi + s\psi\partial_t\alpha - \partial_\nu\psi = (N_2\psi)_1 + (N_2\psi)_2 + (N_2\psi)_3,$$

$$\tilde{f} = f + s\lambda\psi\xi\Delta\eta^0 - s\lambda^2\psi\xi|\nabla\eta^0|^2 + a\psi,$$

$$\tilde{g} = g + b\psi.$$

Applying  $\|\cdot\|_{L^2(\Omega_T)}^2$  resp.  $\|\cdot\|_{L^2(\Gamma_T)}^2$  to the equations in (3.7) and adding the resulting identities, we obtain

$$\begin{aligned} \|\tilde{f}\|_{L^2(\Omega_T)}^2 + \|\tilde{g}\|_{L^2(\Gamma_T)}^2 &= \|M_1\psi\|_{L^2(\Omega_T)}^2 + \|M_2\psi\|_{L^2(\Omega_T)}^2 + \|N_1\psi\|_{L^2(\Gamma_T)}^2 + \|N_2\psi\|_{L^2(\Gamma_T)}^2 \\ &\quad + 2\sum_{i,j} \langle (M_1\psi)_i, (M_2\psi)_j \rangle_{L^2(\Omega_T)} + 2\sum_{i,j} \langle (N_1\psi)_i, (N_2\psi)_j \rangle_{L^2(\Gamma_T)}. \end{aligned} \quad (3.8)$$

*Step 2. Estimating the mixed terms in (3.8) from below.* We often use the following basic pointwise estimates on  $\overline{\Omega}$ ,

$$e^{2\lambda m\|\eta^0\|_\infty} \leq e^{2\lambda(m\|\eta^0\|_\infty + \eta^0)}, \quad |\nabla\alpha| \leq C\lambda\xi, \quad |\partial_t\alpha| \leq C\xi^2, \quad |\partial_t\xi| \leq C\xi^2. \quad (3.9)$$

*Step 2a.* We start with the negative term

$$\langle (M_1\psi)_1, (M_2\psi)_1 \rangle_{L^2(\Omega_T)} = -2s^3\lambda^4 \int_{\Omega_T} |\nabla\eta^0|^4 \xi^3 \psi^2 dx dt.$$

Using integration by parts, (3.3) and (3.1), we further derive

$$\begin{aligned} \langle (M_1\psi)_2, (M_2\psi)_1 \rangle_{L^2(\Omega_T)} &= -s^3\lambda^3 \int_{\Omega_T} |\nabla\eta^0|^2 \xi^3 \nabla\eta^0 \cdot \nabla(\psi^2) dx dt \\ &= s^3\lambda^3 \int_{\Omega_T} \operatorname{div}(|\nabla\eta^0|^2 \xi^3 \nabla\eta^0) \psi^2 dx dt - s^3\lambda^3 \int_{\Gamma_T} |\nabla\eta^0|^2 \xi^3 \partial_\nu \eta^0 \psi^2 dS dt \\ &= 3s^3\lambda^4 \int_{\Omega_T} |\nabla\eta^0|^4 \xi^3 \psi^2 dx dt + s^3\lambda^3 \int_{\Omega_T} \Delta\eta^0 |\nabla\eta^0|^2 \xi^3 \psi^2 dx dt \\ &\quad + s^3\lambda^3 \int_{\Omega_T} (\nabla(|\nabla\eta^0|^2) \cdot \nabla\eta^0) \xi^3 \psi^2 dx dt - s^3\lambda^3 \int_{\Gamma_T} |\partial_\nu \eta^0|^2 \xi^3 \partial_\nu \eta^0 \psi^2 dS dt. \end{aligned}$$

For sufficiently large  $\lambda_1$ , the fact that  $\nabla\eta^0 \neq 0$  on  $\overline{\Omega \setminus \omega'}$  implies

$$\begin{aligned} \langle (M_1\psi)_1, (M_2\psi)_1 \rangle_{L^2(\Omega_T)} + \langle (M_1\psi)_2, (M_2\psi)_1 \rangle_{L^2(\Omega_T)} & \tag{3.10} \\ & \geq Cs^3\lambda^4 \int_{\Omega_T} \xi^3 \psi^2 dx dt - Cs^3\lambda^4 \int_{(0,T) \times \omega'} \xi^3 \psi^2 dx dt - s^3\lambda^3 \int_{\Gamma_T} |\partial_\nu \eta^0|^2 \xi^3 \partial_\nu \eta^0 \psi^2 dS dt \end{aligned}$$

for some  $C = C(\Omega, \omega)$ . Integrating by parts in time, we continue with

$$\begin{aligned} \langle (M_1\psi)_3, (M_2\psi)_1 \rangle_{L^2(\Omega_T)} &= \frac{1}{2}s^2\lambda^2 \int_{\Omega_T} |\nabla\eta^0|^2 \xi^2 \partial_t(\psi^2) dx dt = -s^2\lambda^2 \int_{\Omega_T} |\nabla\eta^0|^2 \partial_t \xi \xi \psi^2 dx dt \\ &\geq -Cs^2\lambda^2 \int_{\Omega_T} \xi^3 \psi^2 dx dt, \end{aligned}$$

where we employed (3.9) and that  $\psi(0) = \psi(T) = 0$ . This term is absorbed by (3.10) for large  $\lambda_1$ . Altogether, we have shown

$$\begin{aligned} \sum_{i=1,2,3} \langle (M_1\psi)_i, (M_2\psi)_1 \rangle_{L^2(\Omega_T)} & \\ & \geq Cs^3\lambda^4 \int_{\Omega_T} \xi^3 \psi^2 dx dt - Cs^3\lambda^4 \int_{(0,T) \times \omega'} \xi^3 \psi^2 dx dt - s^3\lambda^3 \int_{\Gamma_T} |\partial_\nu \eta^0|^2 \xi^3 \partial_\nu \eta^0 \psi^2 dS dt \\ & \geq Cs^3\lambda^4 \int_{\Omega_T} \xi^3 \psi^2 dx dt - Cs^3\lambda^4 \int_{(0,T) \times \omega'} \xi^3 \psi^2 dx dt + Cs^3\lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 dS dt, \end{aligned}$$

using also (3.1). We remark that first and third terms in the last line are the dominant positive terms involving  $\psi^2$ , whereas the second term will lead to a ‘control term’ on the right hand side of the final estimate.

*Step 2b.* Integration by parts and (3.2) yield

$$\begin{aligned} \langle (M_1\psi)_1, (M_2\psi)_2 \rangle_{L^2(\Omega_T)} &= -2s\lambda^2 \int_{\Omega_T} |\nabla\eta^0|^2 \xi \psi \Delta\psi dx dt \\ &= 2s\lambda^2 \int_{\Omega_T} \nabla(|\nabla\eta^0|^2 \xi \psi) \cdot \nabla\psi dx dt - 2s\lambda^2 \int_{\Gamma_T} |\nabla\eta^0|^2 \xi \psi \partial_\nu \psi dS dt \\ &= 2s\lambda^2 \int_{\Omega_T} |\nabla\eta^0|^2 \xi |\nabla\psi|^2 dx dt + 2s\lambda^2 \int_{\Omega_T} \xi \psi \nabla(|\nabla\eta^0|^2) \cdot \nabla\psi dx dt \\ &\quad + 2s\lambda^3 \int_{\Omega_T} |\nabla\eta^0|^2 \psi \xi \nabla\eta^0 \cdot \nabla\psi dx dt - 2s\lambda^2 \int_{\Gamma_T} |\nabla\eta^0|^2 \xi \psi \partial_\nu \psi dS dt. \end{aligned}$$

As above, the first summand will lead to a term controlling  $|\nabla\psi|^2$ . We now apply Young’s inequality to  $(\lambda^2 \xi^{1/2} \psi)(\xi^{1/2} \nabla(|\nabla\eta^0|^2) \cdot \nabla\psi)$  and  $(s\lambda\psi\xi)(\nabla\eta^0 \cdot \nabla\psi)$ , respectively, and estimate

$$\left| 2s\lambda^2 \int_{\Omega_T} \xi \psi \nabla(|\nabla\eta^0|^2) \cdot \nabla\psi dx dt \right| \leq Cs\lambda^4 \int_{\Omega_T} \xi \psi^2 dx dt + Cs \int_{\Omega_T} \xi |\nabla\psi|^2 dx dt,$$

$$\left| 2s\lambda^3 \int_{\Omega_T} |\nabla\eta^0|^2 \psi \xi \nabla\eta^0 \cdot \nabla\psi \, dx \, dt \right| \leq Cs^2\lambda^4 \int_{\Omega_T} \xi^2 \psi^2 \, dx \, dt + C\lambda^2 \int_{\Omega_T} |\nabla\psi|^2 \, dx \, dt.$$

It follows

$$\begin{aligned} \langle (M_1\psi)_1, (M_2\psi)_2 \rangle_{L^2(\Omega_T)} &\geq 2s\lambda^2 \int_{\Omega_T} |\nabla\eta^0|^2 \xi |\nabla\psi|^2 \, dx \, dt - Cs^2\lambda^4 \int_{\Omega_T} \xi^2 \psi^2 \, dx \, dt \\ &\quad - C \int \int_{\Omega_T} (s\xi + \lambda^2) |\nabla\psi|^2 \, dx \, dt - 2s\lambda^2 \int_{\Gamma_T} |\nabla\eta^0|^2 \xi \psi \partial_\nu \psi \, dS \, dt. \end{aligned}$$

The next summand is given by

$$\begin{aligned} \langle (M_1\psi)_2, (M_2\psi)_2 \rangle_{L^2(\Omega_T)} &= -2s\lambda \int_{\Omega_T} (\nabla\eta^0 \cdot \nabla\psi) \xi \Delta\psi \, dx \, dt \\ &= -2s\lambda \int_{\Gamma_T} \nabla\eta^0 \cdot \nabla\psi \partial_\nu \psi \xi \, dS \, dt + 2s\lambda \int_{\Omega_T} \nabla((\nabla\eta^0 \cdot \nabla\psi) \xi) \cdot \nabla\psi \, dx \, dt \\ &= -2s\lambda \int_{\Gamma_T} \nabla\eta^0 \cdot \nabla\psi \partial_\nu \psi \xi \, dS \, dt + 2s\lambda \int_{\Omega_T} \xi (\nabla^2 \eta^0 \nabla\psi) \cdot \nabla\psi \, dx \, dt \quad (3.11) \\ &\quad + 2s\lambda^2 \int_{\Omega_T} \xi |\nabla\eta^0 \cdot \nabla\psi|^2 \, dx \, dt + s\lambda \int_{\Omega_T} \nabla\eta^0 \cdot \nabla |\nabla\psi|^2 \xi \, dx \, dt \end{aligned}$$

due to integration by parts and (3.3). In the sum (3.11), the third term is nonnegative, and the second one can be bounded by

$$2s\lambda \int_{\Omega_T} \xi (\nabla^2 \eta^0 \nabla\psi) \cdot \nabla\psi \, dx \, dt \geq -Cs\lambda \int_{\Omega_T} \xi |\nabla\psi|^2 \, dx \, dt. \quad (3.12)$$

For the last term in (3.11), integration by parts and (3.3) yield

$$\begin{aligned} s\lambda \int_{\Omega_T} \nabla\eta^0 \cdot \nabla |\nabla\psi|^2 \xi \, dx \, dt &= s\lambda \int_{\Gamma_T} \partial_\nu \eta^0 |\nabla\psi|^2 \xi \, dS \, dt - s\lambda \int_{\Omega_T} \operatorname{div}(\xi \nabla\eta^0) |\nabla\psi|^2 \, dx \, dt \\ &= s\lambda \int_{\Gamma_T} \partial_\nu \eta^0 |\nabla\psi|^2 \xi \, dS \, dt - s\lambda^2 \int_{\Omega_T} \xi |\nabla\eta^0|^2 |\nabla\psi|^2 \, dx \, dt - s\lambda \int_{\Omega_T} \xi \cdot \Delta\eta^0 |\nabla\psi|^2 \, dx \, dt. \end{aligned}$$

The last term in the above line can be estimated as in (3.12). We then infer

$$\begin{aligned} \langle (M_1\psi)_2, (M_2\psi)_2 \rangle_{L^2(\Omega_T)} &\geq -2s\lambda \int_{\Gamma_T} \nabla\eta^0 \cdot \nabla\psi \partial_\nu \psi \xi \, dS \, dt - Cs\lambda \int_{\Omega_T} \xi |\nabla\psi|^2 \, dx \, dt \\ &\quad + s\lambda \int_{\Gamma_T} \partial_\nu \eta^0 |\nabla\psi|^2 \xi \, dS \, dt - s\lambda^2 \int_{\Omega_T} \xi |\nabla\eta^0|^2 |\nabla\psi|^2 \, dx \, dt. \end{aligned}$$

Because  $\nabla\psi$  vanishes at  $t = 0$  and  $t = T$  in view of (3.4), we obtain

$$\langle (M_1\psi)_3, (M_2\psi)_2 \rangle_{L^2(\Omega_T)} = \int_{\Omega_T} \partial_t \psi \Delta\psi \, dx \, dt = \int_{\Gamma_T} \partial_t \psi \partial_\nu \psi \, dS \, dt. \quad (3.13)$$

We summarize the inequalities of this step and invoke again that  $\nabla\eta^0 \neq 0$  on  $\overline{\Omega \setminus \omega'}$ . Using also (3.1), we arrive at

$$\begin{aligned} \sum_{i=1,2,3} \langle (M_1\psi)_i, (M_2\psi)_2 \rangle_{L^2(\Omega_T)} &\geq s\lambda^2 \int_{\Omega_T} |\nabla\eta^0|^2 \xi |\nabla\psi|^2 \, dx \, dt - Cs^2\lambda^4 \int_{\Omega_T} \xi^2 \psi^2 \, dx \, dt \\ &\quad - C \int_{\Omega_T} (s\xi + \lambda^2) |\nabla\psi|^2 \, dx \, dt - 2s\lambda^2 \int_{\Gamma_T} |\nabla\eta^0|^2 \xi \psi \partial_\nu \psi \, dS \, dt \\ &\quad - 2s\lambda \int_{\Gamma_T} \nabla\eta^0 \cdot \nabla\psi \partial_\nu \psi \xi \, dS \, dt - Cs\lambda \int_{\Omega_T} \xi |\nabla\psi|^2 \, dx \, dt \\ &\quad + s\lambda \int_{\Gamma_T} \partial_\nu \eta^0 |\nabla\psi|^2 \xi \, dS \, dt + \int_{\Gamma_T} \partial_t \psi \partial_\nu \psi \, dS \, dt \\ &\geq Cs\lambda^2 \int_{\Omega_T} \xi |\nabla\psi|^2 \, dx \, dt - Cs\lambda^2 \int_{(0,T) \times \omega'} \xi |\nabla\psi|^2 \, dx \, dt - Cs^2\lambda^4 \int_{\Omega_T} \xi^2 \psi^2 \, dx \, dt \end{aligned}$$

$$\begin{aligned}
& -2s\lambda^2 \int_{\Gamma_T} (\partial_\nu \eta^0)^2 \xi \psi \partial_\nu \psi \, dS \, dt - 2s\lambda \int_{\Gamma_T} \partial_\nu \eta^0 (\partial_\nu \psi)^2 \xi \, dS \, dt \\
& + s\lambda \int_{\Gamma_T} \partial_\nu \eta^0 |\nabla \psi|^2 \xi \, dS \, dt + \int_{\Gamma_T} \partial_t \psi \partial_\nu \psi \, dS \, dt,
\end{aligned}$$

increasing  $\lambda_1$  and  $s_1$  if necessary.

*Step 2c.* Employing (3.9), we estimate

$$\langle (M_1 \psi)_1, (M_2 \psi)_3 \rangle_{L^2(\Omega_T)} = -2s^2 \lambda^2 \int_{\Omega_T} |\nabla \eta^0|^2 \partial_t \alpha \xi \psi^2 \, dx \, dt \geq -Cs^2 \lambda^2 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt.$$

This term will be absorbed by (3.10) for large  $\lambda$ . Integration by parts, (3.3) and (3.9) next imply

$$\begin{aligned}
\langle (M_1 \psi)_2, (M_2 \psi)_3 \rangle_{L^2(\Omega_T)} &= -s^2 \lambda \int_{\Omega_T} \partial_t \alpha \xi \nabla \eta^0 \cdot \nabla (\psi^2) \, dx \, dt \\
&= -s^2 \lambda \int_{\Gamma_T} \partial_t \alpha \xi \partial_\nu \eta^0 \psi^2 \, dS \, dt + s^2 \lambda \int_{\Omega_T} \operatorname{div}(\partial_t \alpha \xi \nabla \eta^0) \psi^2 \, dx \, dt \\
&= -s^2 \lambda \int_{\Gamma_T} \partial_t \alpha \xi \partial_\nu \eta^0 \psi^2 \, dS \, dt + s^2 \lambda \int_{\Omega_T} \nabla(\partial_t \alpha) \cdot \nabla \eta^0 \xi \psi^2 \, dx \, dt \\
&\quad + s^2 \lambda \int_{\Omega_T} \partial_t \alpha \nabla \xi \cdot \nabla \eta^0 \psi^2 \, dx \, dt + s^2 \lambda \int_{\Omega_T} \partial_t \alpha \xi \Delta \eta^0 \psi^2 \, dx \, dt \\
&\geq -Cs^2 \lambda \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt - Cs^2 \lambda^2 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt.
\end{aligned}$$

Integrating by parts with respect to time, we can derive

$$\begin{aligned}
\langle (M_1 \psi)_3, (M_2 \psi)_3 \rangle_{L^2(\Omega_T)} &= \frac{s}{2} \int_{\Omega_T} \partial_t \alpha \partial_t (\psi^2) \, dx \, dt = -\frac{s}{2} \int_{\Omega_T} \partial_t^2 \alpha \psi^2 \, dx \, dt \\
&\geq -Cs \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt,
\end{aligned} \tag{3.14}$$

since  $\psi$  vanishes at the endpoints and  $|\partial_t^2 \alpha| \leq C\xi^3$ . We conclude from the above inequalities that

$$\sum_{i=1,2,3} \langle (M_1 \psi)_i, (M_2 \psi)_3 \rangle_{L^2(\Omega_T)} \geq -Cs^2 \lambda^2 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt - Cs^2 \lambda \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt.$$

*Step 2d.* We now consider the boundary terms  $N_1$  and  $N_2$ , employing the surface divergence theorem (2.6) several times. We first compute

$$\langle (N_1 \psi)_1, (N_2 \psi)_1 \rangle_{L^2(\Gamma_T)} = \int_{\Gamma_T} \delta \partial_t \psi \Delta_\Gamma \psi \, dS \, dt = -\frac{\delta}{2} \int_{\Gamma_T} \partial_t |\nabla_\Gamma \psi|^2 \, dS \, dt = 0$$

by means of  $\psi(0) = \psi(T) = 0$ . Moreover, (3.2) yields

$$\begin{aligned}
\langle (N_1 \psi)_2, (N_2 \psi)_1 \rangle_{L^2(\Gamma_T)} &= \delta s \lambda \int_{\Gamma_T} \partial_\nu \eta^0 \xi \Delta_\Gamma \psi \, dS \, dt = -\delta s \lambda \int_{\Gamma_T} \langle \nabla_\Gamma (\partial_\nu \eta^0 \xi \psi), \nabla_\Gamma \psi \rangle_\Gamma \, dS \, dt \\
&= -\delta s \lambda \int_{\Gamma_T} \xi \psi \langle \nabla_\Gamma \partial_\nu \eta^0, \nabla_\Gamma \psi \rangle_\Gamma \, dS \, dt - \delta s \lambda \int_{\Gamma_T} \partial_\nu \eta^0 \xi |\nabla_\Gamma \psi|^2 \, dS \, dt.
\end{aligned}$$

The next two terms are estimated by

$$\begin{aligned}
\langle (N_1 \psi)_1, (N_2 \psi)_2 \rangle_{L^2(\Gamma_T)} &= \frac{s}{2} \int_{\Gamma_T} \partial_t \alpha \partial_t (\psi^2) \, dS \, dt \geq -Cs \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt, \\
\langle (N_1 \psi)_2, (N_2 \psi)_2 \rangle_{L^2(\Gamma_T)} &= s^2 \lambda \int_{\Gamma_T} \partial_\nu \eta^0 \partial_t \alpha \xi \psi^2 \, dS \, dt \geq -Cs^2 \lambda \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt,
\end{aligned}$$

where we proceed as in (3.14) and use (3.9), respectively. Finally, the summand

$$\langle (N_1 \psi)_1, (N_2 \psi)_3 \rangle_{L^2(\Gamma_T)} = -\int_{\Gamma_T} \partial_t \psi \partial_\nu \psi \, dS \, dt$$

cancels with the one from (3.13), and

$$\langle (N_1\psi)_2, (N_2\psi)_3 \rangle_{L^2(\Gamma_T)} = -s\lambda \int_{\Gamma_T} \xi \partial_\nu \eta^0 \partial_\nu \psi \psi \, dS \, dt.$$

*Step 3. The transformed estimate.* We collect the final inequalities in Steps 2a–2d. Increasing  $\lambda_1$  and  $s_1$  if needed to absorb lower order terms, we arrive at

$$\begin{aligned}
 & \sum_{i,j} \langle (M_1\psi)_i, (M_2\psi)_j \rangle_{L^2(\Omega_T)} + \sum_{i,j} \langle (N_1\psi)_i, (N_2\psi)_j \rangle_{L^2(\Gamma_T)} \\
 & \geq Cs^3\lambda^4 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt - Cs^3\lambda^4 \int_{(0,T) \times \omega'} \xi^3 \psi^2 \, dx \, dt \\
 & \quad + Cs^3\lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt + Cs\lambda^2 \int_{\Omega_T} \xi |\nabla \psi|^2 \, dx \, dt \\
 & \quad - Cs\lambda^2 \int_{(0,T) \times \omega'} \xi |\nabla \psi|^2 \, dx \, dt - 2s\lambda^2 \int_{\Gamma_T} (\partial_\nu \eta^0)^2 \xi \psi \partial_\nu \psi \, dS \, dt \\
 & \quad - 2s\lambda \int_{\Gamma_T} \partial_\nu \eta^0 (\partial_\nu \psi)^2 \xi \, dS \, dt + s\lambda \int_{\Gamma_T} \partial_\nu \eta^0 |\nabla \psi|^2 \xi \, dS \, dt \\
 & \quad - \delta s\lambda \int_{\Gamma_T} \xi \psi \langle \nabla_\Gamma \partial_\nu \eta^0, \nabla_\Gamma \psi \rangle_\Gamma \, dS \, dt - \delta s\lambda \int_{\Gamma_T} \partial_\nu \eta^0 \xi |\nabla_\Gamma \psi|^2 \, dS \, dt \\
 & \quad - s\lambda \int_{\Gamma_T} \xi \partial_\nu \eta^0 \partial_\nu \psi \psi \, dS \, dt.
 \end{aligned}$$

We combine this estimate with (3.8). The expressions for  $\tilde{f}$  and  $\tilde{g}$  lead to additional lower order terms which can be absorbed to the left-hand side for large  $\lambda_1$  and  $s_1$ . Using also  $|\nabla \psi|^2 = |\nabla_\Gamma \psi|^2 + |\partial_\nu \psi|^2$  and (3.1), we deduce

$$\begin{aligned}
 & \|M_1\psi\|_{L^2(\Omega_T)}^2 + \|M_2\psi\|_{L^2(\Omega_T)}^2 + \|N_1\psi\|_{L^2(\Gamma_T)}^2 + \|N_2\psi\|_{L^2(\Gamma_T)}^2 + s^3\lambda^4 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt \\
 & \quad + s\lambda^2 \int_{\Omega_T} \xi |\nabla \psi|^2 \, dx \, dt + s^3\lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt + s\lambda \int_{\Gamma_T} \xi |\nabla_\Gamma \psi|^2 \, dS \, dt + s\lambda \int_{\Gamma_T} \xi (\partial_\nu \psi)^2 \, dS \, dt \\
 & \leq C \int_{\Omega_T} e^{-2s\alpha} |\partial_t \varphi + \Delta \varphi - a\varphi|^2 \, dx \, dt + C \int_{\Gamma_T} e^{-2s\alpha} |\partial_t \varphi + \delta \Delta_\Gamma \varphi_\Gamma - \partial_\nu \varphi - b\varphi|^2 \, dS \, dt \\
 & \quad + Cs^3\lambda^4 \int_{(0,T) \times \omega'} \xi^3 \psi^2 \, dx \, dt + Cs\lambda^2 \int_{(0,T) \times \omega'} \xi |\nabla \psi|^2 \, dx \, dt \\
 & \quad + Cs\lambda^2 \int_{\Gamma_T} (\partial_\nu \eta^0)^2 \xi |\psi| |\partial_\nu \psi| \, dS \, dt + Cs\lambda \int_{\Gamma_T} \xi |\partial_\nu \eta^0| |\nabla_\Gamma \psi|^2 \, dS \, dt \\
 & \quad + Cs\lambda \int_{\Gamma_T} \xi \psi |\nabla_\Gamma \partial_\nu \eta^0| |\nabla_\Gamma \psi| \, dS \, dt + Cs\lambda \int_{\Gamma_T} \xi |\partial_\nu \eta^0| |\partial_\nu \psi| |\psi| \, dS \, dt. \tag{3.15}
 \end{aligned}$$

We denote the four latter boundary integrals on the right-hand side of (3.15) by  $I_1, \dots, I_4$ . Young's inequality allows to estimate  $I_1$  by

$$\begin{aligned}
 I_1 & = Cs \int_{\Gamma_T} |\lambda^{3/2} \xi^{1/2} \psi| |\lambda^{1/2} \xi^{1/2} \partial_\nu \psi| \, dS \, dt \\
 & \leq C \left( s^2 \lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt + \lambda \int_{\Gamma_T} \xi (\partial_\nu \psi)^2 \, dS \, dt \right). \tag{3.16}
 \end{aligned}$$

Increasing  $s_1$  if necessary, we can then control (3.16) by the left-hand side of (3.15). The fourth boundary integral  $I_4$  is treated analogously. For the integral  $I_3$ , we have

$$I_3 \leq C \left( \int_{\Gamma_T} \xi |\nabla_\Gamma \psi|^2 \, dS \, dt + s^2 \lambda^2 \int_{\Gamma_T} \xi \psi^2 \, dS \, dt \right). \tag{3.17}$$

Increasing again  $s_1$  and  $\lambda_1$  if necessary, (3.17) can be absorbed by the left-hand side of (3.15).

For the last integral  $I_2$ , we use the identity (2.6) and that  $\xi(t, \cdot)$  is constant on  $\Gamma$ . We then obtain

$$\begin{aligned} I_2 &\leq Cs\lambda \int_0^T \int_{\Gamma} \xi |\nabla_{\Gamma} \psi|^2 dS dt \\ &\leq C \int_0^T \int_{\Gamma} (s^{-1/2} \xi^{-1/2} |\Delta_{\Gamma} \psi|) (s^{3/2} \lambda \xi^{3/2} |\psi|) dS dt \\ &\leq s^{-1} \int_{\Gamma_T} \xi^{-1} |\Delta_{\Gamma} \psi|^2 dS dt + Cs^3 \lambda^2 \int_{\Gamma_T} \xi^3 \psi^2 dS dt. \end{aligned} \quad (3.18)$$

The second summand in (3.18) can be absorbed by the left-hand side of (3.15) choosing a sufficiently large  $\lambda_1$ . Altogether, we thus arrive at

$$\begin{aligned} &\|M_1 \psi\|_{L^2(\Omega_T)}^2 + \|M_2 \psi\|_{L^2(\Omega_T)}^2 + \|N_1 \psi\|_{L^2(\Gamma_T)}^2 + \|N_2 \psi\|_{L^2(\Gamma_T)}^2 + s^3 \lambda^4 \int_{\Omega_T} \xi^3 \psi^2 dx dt \\ &\quad + s\lambda^2 \int_{\Omega_T} \xi |\nabla \psi|^2 dx dt + s^3 \lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 dS dt + s\lambda \int_{\Gamma_T} \xi |\nabla_{\Gamma} \psi|^2 dS dt + s\lambda \int_{\Gamma_T} \xi (\partial_{\nu} \psi)^2 dS dt \\ &\leq C \int_{\Omega_T} e^{-2s\alpha} |\partial_t \varphi + \Delta \varphi - a\varphi|^2 dx dt + C \int_{\Gamma_T} e^{-2s\alpha} |\partial_t \varphi + \delta \Delta_{\Gamma} \varphi - d\partial_{\nu} \varphi - b\varphi|^2 dS dt \\ &\quad + Cs^3 \lambda^4 \int_{\omega'_T} \xi^3 \psi^2 dx dt + Cs\lambda^2 \int_{\omega'_T} \xi |\nabla \psi|^2 dx dt + s^{-1} \int_{\Gamma_T} \xi^{-1} |\Delta_{\Gamma} \psi|^2 dS dt. \end{aligned} \quad (3.19)$$

To put the last summand in (3.19) to the left, we observe that  $\delta \Delta_{\Gamma} \psi = N_2 \psi - s\psi \partial_t \alpha + \partial_{\nu} \psi$ . Combined with (3.9), this identity yields

$$I := s^{-1} \int_{\Gamma_T} \xi^{-1} |\Delta_{\Gamma} \psi|^2 dS dt \leq \frac{1}{2} \|N_2 \psi\|_{L^2(\Gamma_T)}^2 + Cs \int_{\Gamma_T} \xi^3 \psi^2 dS dt + C \int_{\Gamma_T} \xi (\partial_{\nu} \psi)^2 dS dt. \quad (3.20)$$

for sufficiently large  $s_1$ . We can now choose sufficiently large  $\lambda_1$  and  $s_1$  so that (3.19) becomes

$$\begin{aligned} &\|M_1 \psi\|_{L^2(\Omega_T)}^2 + \|M_2 \psi\|_{L^2(\Omega_T)}^2 + \|N_1 \psi\|_{L^2(\Gamma_T)}^2 + \|N_2 \psi\|_{L^2(\Gamma_T)}^2 + s^3 \lambda^4 \int_{\Omega_T} \xi^3 \psi^2 dx dt \\ &\quad + s\lambda^2 \int_{\Omega_T} \xi |\nabla \psi|^2 dx dt + s^3 \lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 dS dt + s\lambda \int_{\Gamma_T} \xi (\partial_{\nu} \psi)^2 dS dt + s\lambda \int_{\Gamma_T} \xi |\nabla_{\Gamma} \psi|^2 dS dt \\ &\leq C \int_{\Omega_T} e^{-2s\alpha} |\partial_t \varphi + \Delta \varphi - a\varphi|^2 dx dt + C \int_{\Gamma_T} e^{-2s\alpha} |\partial_t \varphi + \delta \Delta_{\Gamma} \varphi - \partial_{\nu} \varphi - b\varphi|^2 dS dt \\ &\quad + Cs^3 \lambda^4 \int_{\omega'_T} \xi^3 \psi^2 dx dt + Cs\lambda^2 \int_{\omega'_T} \xi |\nabla \psi|^2 dx dt. \end{aligned} \quad (3.21)$$

*Step 4. Inverting the transformation.* The inequality (3.20) allows to replace in (3.21) the summand  $\|N_2 \psi\|_{L^2(\Gamma_T)}^2$  by the term  $I$  times a constant, where we increase  $\lambda_1$  and  $s_1$  if necessary to absorb the lower order terms in (3.20). Similarly, from  $\partial_t \psi = N_1 \psi - s\lambda \psi \xi \partial_{\nu} \eta^0$ , we deduce that

$$I' := s^{-1} \int_{\Gamma_T} \xi^{-1} |\partial_t \psi|^2 dS dt \leq C \|N_1 \psi\|_{L^2(\Gamma_T)}^2 + Cs\lambda^2 \int_{\Gamma_T} \xi \psi^2 dS dt,$$

and hence also  $I'$  can be put on the left hand side of (3.21). In a similar way one handles the corresponding terms on  $\Omega_T$ , see (1.58) and (1.59) in [11]. We thus infer

$$\begin{aligned} &s^{-1} \int_{\Omega_T} \xi^{-1} (|\partial_t \psi|^2 + |\Delta \psi|^2) dx dt + s^{-1} \int_{\Gamma_T} \xi^{-1} (|\partial_t \psi|^2 + |\Delta_{\Gamma} \psi|^2) dS dt \\ &\quad + s^3 \lambda^4 \int_{\Omega_T} \xi^3 \psi^2 dx dt + s\lambda^2 \int_{\Omega_T} \xi |\nabla \psi|^2 dx dt \\ &\quad + s^3 \lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 dS dt + s\lambda \int_{\Gamma_T} \xi (\partial_{\nu} \psi)^2 dS dt + s\lambda \int_{\Gamma_T} \xi |\nabla_{\Gamma} \psi|^2 dS dt \\ &\leq C \int_{\Omega_T} e^{-2s\alpha} |\partial_t \varphi + \Delta \varphi - a\varphi|^2 dx dt + C \int_{\Gamma_T} e^{-2s\alpha} |\partial_t \varphi + \delta \Delta_{\Gamma} \varphi - \partial_{\nu} \varphi - b\varphi|^2 dS dt \end{aligned} \quad (3.22)$$

$$+ Cs^3\lambda^4 \int_{\omega_T} \xi^3 \psi^2 dx dt + Cs\lambda^2 \int_{\omega'_T} \xi |\nabla \psi|^2 dx dt,$$

also using  $\omega' \subset \omega$ . As on p.1409 of [11], one can now absorb the gradient term on the right-hand side by the integral on  $\omega_T$  and the left-hand side.

It remains to insert  $\psi = e^{-s\alpha}\varphi$  into (3.22). The terms involving derivatives of  $\psi$  then lead to various lower order terms which can be controlled by the other terms in (3.22). For summands on  $\Omega_T$  this is done in Step 3 of the proof of Lemma 1.3 in [11]. Since the new terms on the boundary can be treated in the same way, we omit the details. One thus obtains the asserted Carleman estimate for the original function  $\varphi$ .  $\square$

*Remark 3.3.* Up to inequality (3.15), the arguments in the above proof remain valid also for  $\delta = 0$ . However, in our proof the assumption  $\delta > 0$  is essential to put the third and fourth boundary integral  $I_3$  and  $I_4$  on the right-hand side of (3.15) to the left.

**4. Null controllability.** In this section we apply the Carleman estimate to show null controllability for (1.1) and its generalizations. Throughout we fix  $T > 0$ ,  $\omega \Subset \Omega$ ,  $d, \delta > 0$ ,  $a \in L^\infty(Q_T)$  and  $b \in L^\infty(\Gamma_T)$ . We assume that  $s_1$  and  $\lambda_1$  are sufficiently large to apply Lemma 3.2 for this data.

**4.1. Interior control for linear equations.** We show the null controllability of the linear inhomogeneous system

$$\partial_t y - d\Delta y + a(t, x)y = v(t, x)\mathbb{1}_\omega + f(t, x) \quad \text{in } \Omega_T, \quad (4.1)$$

$$\partial_t y_\Gamma - \delta\Delta_\Gamma y + d\partial_\nu y + b(t, x)y_\Gamma = g(t, x) \quad \text{on } \Gamma_T, \quad (4.2)$$

$$y_\Gamma(t, x) = y|_\Gamma(t, x) \quad \text{on } \Gamma_T, \quad (4.3)$$

$$(y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}) \quad \text{in } \Omega \times \Gamma, \quad (4.4)$$

by a standard duality argument. To this end, we first derive an observability estimate for the homogeneous backward system

$$-\partial_t \varphi - d\Delta \varphi + a(t, x)\varphi = 0 \quad \text{in } \Omega_T, \quad (4.5)$$

$$-\partial_t \varphi_\Gamma - \delta\Delta_\Gamma \varphi_\Gamma + d\partial_\nu \varphi + b(t, x)\varphi_\Gamma = 0 \quad \text{on } \Gamma_T, \quad (4.6)$$

$$\varphi_\Gamma(t, x) = \varphi|_\Gamma(t, x) \quad \text{on } \Gamma_T, \quad (4.7)$$

$$(\varphi(T, \cdot), \varphi_\Gamma(T, \cdot)) = (\varphi_T, \varphi_{T,\Gamma}) \quad \text{in } \Omega \times \Gamma, \quad (4.8)$$

from the Carleman estimate. In addition, we can easily infer the final state observability of the system (4.1)–(4.4) with  $f = g = v = 0$ , which is of independent interest.

**Proposition 4.1.** *There is a constant  $C > 0$  such that for all  $(\varphi_T, \varphi_{T,\Gamma}) \in \mathbb{L}^2$  the mild solution  $(\varphi, \varphi_\Gamma)$  of the backward problem (4.5)–(4.8) satisfies*

$$\|\varphi(0, \cdot)\|^2 + \|\varphi_\Gamma(0, \cdot)\|^2 \leq C \int_{\omega_T} |\varphi|^2 dx dt. \quad (4.9)$$

Moreover, for  $(y_0, y_{0,\Gamma}) \in \mathbb{L}^2$  the mild solution  $(y, y_\Gamma)$  of the forward problem (4.1)–(4.4) with  $f = g = v = 0$  satisfies

$$\|y(T, \cdot)\|^2 + \|y_\Gamma(T, \cdot)\|^2 \leq C \int_{\omega_T} |y|^2 dx dt. \quad (4.10)$$

Given  $R > 0$ , the constants  $C = C(R)$  can be chosen independently of all  $a, b$  with  $\|a\|_\infty, \|b\|_\infty \leq R$ .

*Proof.* We argue as in Section 1.2 of [11]. Recall from Subsection 2.4 that the backward problem (4.5)–(4.8) possesses a unique mild solution  $(\varphi, \varphi_\Gamma) \in C([0, T]; \mathbb{L}^2)$  which depends continuously in  $\mathbb{L}^2$  on  $(\varphi_T, \varphi_{T,\Gamma})$ . By density we can thus restrict ourselves to final values  $(\varphi_T, \varphi_{T,\Gamma}) \in \mathbb{H}^1$ , so

that  $(\varphi_T, \varphi_{T,\Gamma}) \in \mathbb{E}_1$  is a strong solution due to the backward version of Proposition 2.5. For fixed  $\lambda = \lambda_1$  and  $s = s_1$ , the Carleman estimate from Lemma 3.2 thus implies

$$\iint_{(T/4, 3T/4) \times \Omega} |\varphi|^2 dx dt + \iint_{(T/4, 3T/4) \times \Gamma} |\varphi_\Gamma|^2 dS dt \leq C \int_{\omega_T} |\varphi|^2 dx dt.$$

On the other hand, the backward version of (2.20) on the time interval  $[0, t]$  yields  $\|\varphi(0, \cdot)\|^2 + \|\varphi_\Gamma(0, \cdot)\|^2 \leq C\|\varphi(t, \cdot)\|^2 + \|\varphi_\Gamma(t, \cdot)\|^2$  for all  $t \in (0, T)$ . Integrating this inequality over  $(T/4, 3T/4)$ , we infer

$$\|\varphi(0, \cdot)\|^2 + \|\varphi_\Gamma(0, \cdot)\|^2 \leq C \int_{T/4}^{3T/4} (\|\varphi(t, \cdot)\|^2 + \|\varphi_\Gamma(t, \cdot)\|^2) dt,$$

which yields inequality (4.9). The second assertion then follows by a simple transformation as explained in Subsection 2.4.  $\square$

We now establish the null controllability of the linear system, where we allow for inhomogeneities with exponential decay at  $t = 0$  and  $t = T$ . To this end, we introduce the weighted  $L^2$ -spaces

$$\begin{aligned} Z_\Omega &= \{f \in L^2(\Omega_T) : e^{s\alpha} \xi^{-3/2} f \in L^2(\Omega_T)\}, & \langle f_1, f_2 \rangle_{Z_\Omega} &= \int_{\Omega_T} f_1 f_2 e^{2s\alpha} \xi^{-3} dx dt, \\ Z_\Gamma &= \{g \in L^2(\Gamma_T) : e^{s\alpha} \xi^{-3/2} g \in L^2(\Gamma_T)\}, & \langle g_1, g_2 \rangle_{Z_\Gamma} &= \int_{\Gamma_T} g_1 g_2 e^{2s\alpha} \xi^{-3} dS dt. \end{aligned}$$

In Proposition 4.3 we weaken the assumptions on  $f$  and  $g$ , requiring decay only at  $t = T$ .

**Theorem 4.2.** *For all data  $Y_0 := (y_0, y_{0,\Gamma}) \in \mathbb{L}^2$ ,  $f \in Z_\Omega$  and  $g \in Z_\Gamma$ , there is a control  $v \in L^2(\omega_T)$  such that the mild solution  $(y, y_\Gamma)$  of (4.1)–(4.4) satisfies  $y(T, \cdot) = y_\Gamma(T, \cdot) = 0$ .*

*Proof.* By a duality argument, the assertion will follow from the observability estimate of the previous proposition. To this aim, we define the bounded linear operator  $\mathcal{T} : L^2(\omega_T) \rightarrow \mathbb{L}^2$  by

$$\mathcal{T}v = \int_0^T S(T, \tau)(-\mathbb{1}_\omega v(\tau), 0) d\tau,$$

where the solution operators  $S(T, \tau)$  are given by Proposition 2.5. Using that  $Z_\Omega \times Z_\Gamma \hookrightarrow L^2(\Omega_T) \times L^2(\Gamma_T)$ , we also introduce the bounded linear operator  $\mathcal{S} : \mathbb{L}^2 \times Z_\Omega \times Z_\Gamma \rightarrow \mathbb{L}^2$  by

$$\mathcal{S}(Y_0, f, g) = S(T, 0)Y_0 + \int_0^T S(T, \tau)(f(\tau), g(\tau)) d\tau.$$

Due to Proposition 2.5(a), the function  $\mathcal{S}(Y_0, f, g) - \mathcal{T}v$  is the final value  $(y(T), y_\Gamma(T))$  of the mild solution  $(y, y_\Gamma)$  to the inhomogeneous system (4.1)–(4.4). Note that the adjoint  $\mathcal{T}^* : \mathbb{L}^2 \rightarrow L^2(\omega_T)$  is given by

$$\mathcal{T}^*(\varphi_T, \varphi_{T,\Gamma}) = -\mathbb{1}_\omega \varphi,$$

where  $(\varphi, \varphi_\Gamma) = S(T, \cdot)^*(\varphi_T, \varphi_{T,\Gamma})$  is the mild solution of the homogeneous backward problem (4.5)–(4.8) with final value  $(\varphi_T, \varphi_{T,\Gamma})$ , see Proposition 2.5(f). Taking the weights into account, one determines the adjoint  $\mathcal{S}^* : \mathbb{L}^2 \rightarrow \mathbb{L}^2 \times Z_\Omega \times Z_\Gamma$  as

$$\mathcal{S}^*(\varphi_T, \varphi_{T,\Gamma}) = ((\varphi(0), \varphi_\Gamma(0)), e^{-2s\alpha} \xi^3 \varphi, e^{-2s\alpha} \xi^3 \varphi|_\Gamma).$$

Recall from Subsection 2.4 that  $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$  is a strong solution of the backward problem if  $(\varphi_T, \varphi_{T,\Gamma}) \in \mathbb{H}^1$ . The observability estimate from Proposition 4.1 and the Carleman estimate from Lemma 3.2 now imply that

$$\begin{aligned} \|\mathcal{S}^*(\varphi_T, \varphi_{T,\Gamma})\|_{\mathbb{L}^2 \times Z_\Omega \times Z_\Gamma}^2 &= \|\varphi(0, \cdot)\|^2 + \|\varphi_\Gamma(0, \cdot)\|^2 + \int_{\Omega_T} e^{-2s\alpha} \xi^3 \varphi^2 dx dt + \int_{\Gamma_T} e^{-2s\alpha} \xi^3 \varphi^2 dS dt \\ &\leq C \int_{\omega_T} |\varphi|^2 dx dt = C \|\mathcal{T}^*(\varphi_T, \varphi_{T,\Gamma})\|_{L^2(\omega_T)}, \end{aligned}$$

at first for  $(\varphi_T, \varphi_{T,\Gamma}) \in \mathbb{H}^1$ , but then for  $(\varphi_T, \varphi_{T,\Gamma}) \in \mathbb{L}^2$  by approximation. Theorem IV.2.2 of [35] thus shows that the range of  $\mathcal{T}$  contains that of  $\mathcal{S}$ ; i.e., for all  $Y_0 \in \mathbb{L}^2$ ,  $f \in Z_\Omega$  and  $g \in Z_\Gamma$  there is

a control  $v \in L^2(\omega_T)$  such that  $\mathcal{S}(Y_0, f, g) = \mathcal{T}v$ . Therefore,  $(y(T), y_\Gamma(T)) = \mathcal{S}(Y_0, f, g) - \mathcal{T}v = 0$ , as asserted.  $\square$

The proof of null controllability in the semilinear case is based on a fixed point argument involving a continuous operator mapping the initial value to a null control of the inhomogeneous linear problem. To obtain such an operator, we have to single out a special control. As in Theorem 2.1 of [19] we choose a control having a minimal norm. We prefer to state this more technical point in a proposition below, separated from Theorem 4.2.

To that purpose, for given  $a \in L^\infty(\Omega_T)$  and  $b \in L^\infty(\Gamma_T)$  we define the backward parabolic operator

$$L^* = \begin{pmatrix} -\partial_t - d\Delta + a & 0 \\ d\partial_\nu & -\partial_t - \delta\Delta_\Gamma + b \end{pmatrix} : \mathbb{E}_1 \rightarrow L^2(0, T; \mathbb{L}^2).$$

As in [19] we consider the weights

$$\tilde{\alpha}(t, x) = \frac{t}{T} \alpha(t, x), \quad \tilde{\xi}(t, x) = \frac{t}{T} \xi(t, x), \quad t \in (0, T), \quad x \in \bar{\Omega},$$

which only blow up at  $t = T$ . We introduce the weighted spaces

$$\begin{aligned} \tilde{X} &:= \{Y \in L^2(0, T; \mathbb{L}^2) : e^{s\tilde{\alpha}}Y \in L^2(0, T; \mathbb{L}^2)\}, \\ \tilde{Z}_\Omega &:= \{f \in L^2(\Omega_T) : e^{s\tilde{\alpha}}\tilde{\xi}^{-3/2}f \in L^2(\Omega_T)\}, \\ \tilde{Z}_\Gamma &:= \{g \in L^2(\Gamma_T) : e^{s\tilde{\alpha}}\tilde{\xi}^{-3/2}g \in L^2(\Gamma_T)\}, \end{aligned}$$

endowed with the corresponding scalar products as above. Observe that the weights force decay only at  $t = T$ . We further define the functional  $J$  by

$$J : \tilde{X} \times L^2(\omega_T) \rightarrow \mathbb{R}, \quad J(Y, v) := \frac{1}{2} \int_0^T \|e^{s\tilde{\alpha}(t, \cdot)}Y(t, \cdot)\|_{\mathbb{L}^2}^2 dt + \frac{1}{2} \int_{\omega_T} v^2 dx dt.$$

The general strategy of the following proof is the same as in [19], even though we have not been able to obtain the Lagrange multiplier as in (2.5)–(2.6) of [19]. Hence our arguments differ in certain important points from [19], and we thus give all the details.

**Proposition 4.3.** *Let  $Y_0 := (y_0, y_{0,\Gamma}) \in \mathbb{L}^2$ ,  $f \in \tilde{Z}_\Omega$  and  $g \in \tilde{Z}_\Gamma$ . Then the functional  $J$  has a unique minimizer  $(Y := (y, y_\Gamma), v) \in \tilde{X} \times L^2(\omega_T)$  on the set*

$$M = \{(Y, v) \in \tilde{X} \times L^2(\omega_T) : (Y, v) \text{ is a distributional solution of (4.1)–(4.4) with } Y(T, \cdot) = 0\}.$$

There is a constant  $C > 0$  such that for all  $Y_0 \in \mathbb{L}^2$ ,  $f \in \tilde{Z}_\Omega$  and  $g \in \tilde{Z}_\Gamma$  we have

$$\|Y\|_{\tilde{X}} + \|v\|_{L^2(\omega_T)} \leq C(\|Y_0\|_{\mathbb{L}^2} + \|f\|_{\tilde{Z}_\Omega} + \|g\|_{\tilde{Z}_\Gamma}). \quad (4.11)$$

Given  $R > 0$ , the constant  $C = C(R)$  can be chosen independently of all  $a, b$  with  $\|a\|_\infty, \|b\|_\infty \leq R$ .

*Proof.* Since  $M$  is a closed convex subset of  $X \times L^2(\omega_T)$  and  $J^{1/2}$  is an equivalent norm of  $\tilde{X} \times L^2(\omega_T)$ , the functional  $J$  has a unique minimizer for given  $Y_0, f$  and  $g$  if we can show that the set  $M$  is nonempty. We will construct a function  $(Y_*, v_*) \in M$  as a weak limit of minimizers of regularized problems.

*Step 1.* We first take  $f \in \tilde{Z}_\Omega$  and  $g \in \tilde{Z}_\Gamma$  supported in  $(0, T]$ . Hence,  $f \in Z_\Omega$  and  $g \in Z_\Gamma$ . For each  $\varepsilon \in (0, 1]$ , we introduce a new weight

$$\rho_\varepsilon(t, x) = e^{s\tilde{\alpha}(t, x) \frac{T-t}{T-t+\varepsilon}}, \quad t \in (0, T), \quad x \in \bar{\Omega}$$

which is bounded and strictly positive on  $[0, T] \times \bar{\Omega}$ . We define the corresponding functional  $J_\varepsilon : L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T) \rightarrow \mathbb{R}$  by

$$J_\varepsilon(Y, v) := \frac{1}{2} \int_0^T \|\rho_\varepsilon(t, \cdot)Y(t, \cdot)\|_{\mathbb{L}^2}^2 dt + \frac{1}{2} \int_{\omega_T} v^2 dx dt.$$

We further consider the map  $G : L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T) \rightarrow \mathbb{E}_1^*$  given by

$$\begin{aligned} G(Y, v)[(\varphi, \varphi_\Gamma)] &= \langle Y, L^*(\varphi, \varphi_\Gamma) \rangle_{L^2(0, T; \mathbb{L}^2)} - \langle Y_0, (\varphi(0), \varphi_\Gamma(0)) \rangle_{\mathbb{L}^2} \\ &\quad - \langle f, (\varphi, \varphi_\Gamma) \rangle_{L^2(\Omega_T)} - \langle g, (\varphi, \varphi_\Gamma) \rangle_{L^2(\Gamma_T)} - \langle v, (\varphi, \varphi_\Gamma)(\varphi, \varphi_\Gamma) \rangle_{L^2(\omega_T)} \end{aligned}$$

for  $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$ . We point out that  $(Y, v) \in L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T)$  is a distributional solution of (4.1)–(4.4) with  $Y(T, \cdot) = 0$  if and only if  $G(Y, v) = 0$ . Observe that  $J_\varepsilon^{1/2}$  defines an equivalent norm on  $L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T)$  and that  $\ker G = \{(Y, v) \in L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T) : G(Y, v) = 0\}$  is a closed convex subset of  $L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T)$ . This subset is nonempty by Theorem 4.2, since  $f \in Z_\Omega$  and  $g \in Z_\Gamma$ . Hence,  $J_\varepsilon$  has a unique minimizer  $(Y_\varepsilon, v_\varepsilon)$  on  $\ker G$  for every  $\varepsilon \in (0, 1]$ .

*Step 2.* We show that the functions  $(\rho_\varepsilon Y_\varepsilon, v_\varepsilon)$  are uniformly bounded in  $L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T)$  for  $\varepsilon \in (0, 1]$ . To this end, we note that each functional  $J_\varepsilon$  is continuously differentiable and its derivative at  $(Y, v)$  is given by

$$J'_\varepsilon(Y, v)[Z, u] = \langle \rho_\varepsilon^2 Y, Z \rangle_{L^2(0, T; \mathbb{L}^2)} + \langle v, u \rangle_{L^2(\omega_T)}, \quad (Z, u) \in L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T).$$

The map  $G$  is affine and continuous. Its derivative is the linear operator  $G' : L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T) \rightarrow \mathbb{E}_1^*$  acting as

$$G'(Y, v)[(\varphi, \varphi_\Gamma)] = \langle Y, L^*(\varphi, \varphi_\Gamma) \rangle_{L^2(0, T; \mathbb{L}^2)} - \langle v, (\varphi, \varphi_\Gamma) \rangle_{L^2(\omega_T)}$$

for  $(Y, v) \in L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T)$  and  $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$ . For all  $(Z, u) \in \ker G'$  and  $\sigma \in \mathbb{R}$  we have  $J_\varepsilon(y_\varepsilon, v_\varepsilon) \leq J_\varepsilon((Y_\varepsilon, v_\varepsilon) + \sigma(Z, u))$ , since  $(Y_\varepsilon, v_\varepsilon)$  is the minimizer on  $\ker G$ . At the minimum  $\sigma = 0$ , we derive that  $J'_\varepsilon(Y_\varepsilon, v_\varepsilon)[Z, u] = 0$  for all  $(Z, u) \in \ker G'$ . Therefore

$$(\rho_\varepsilon^2 Y_\varepsilon, v_\varepsilon) \in (\ker G')^\perp = \overline{\text{range}(G')^*}.$$

The adjoint  $(G')^* : \mathbb{E}_1 \rightarrow L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T)$  is given by  $(G')^*(\varphi, \varphi_\Gamma) = (L^*(\varphi, \varphi_\Gamma), -\mathbb{1}_\omega \varphi)$ . Thus there exists a sequence  $(\Phi_n)_n := ((\varphi_n, \varphi_{n, \Gamma}))_n$  in  $\mathbb{E}_1$  such that

$$L^* \Phi_n \rightarrow \rho_\varepsilon^2 Y_\varepsilon \text{ in } L^2(0, T; \mathbb{L}^2) \quad \text{and} \quad -\mathbb{1}_\omega \varphi_n \rightarrow v_\varepsilon \text{ in } L^2(\omega_T),$$

as  $n \rightarrow \infty$ . Since  $(Y_\varepsilon, v_\varepsilon)$  is a distributional solution of (4.1)–(4.4) with  $Y_\varepsilon(T, \cdot) = 0$ , we obtain

$$\begin{aligned} \|\rho_\varepsilon Y_\varepsilon\|_{L^2(0, T; \mathbb{L}^2)}^2 + \|v_\varepsilon\|_{L^2(\omega_T)}^2 &= \langle Y_\varepsilon, \rho_\varepsilon^2 Y_\varepsilon \rangle_{L^2(0, T; \mathbb{L}^2)} + \langle v_\varepsilon, v_\varepsilon \rangle_{L^2(\omega_T)} \\ &= \lim_{n \rightarrow \infty} (\langle Y_\varepsilon, L^* \Phi_n \rangle_{L^2(0, T; \mathbb{L}^2)} - \langle v_\varepsilon, \varphi_n \rangle_{L^2(\omega_T)}) \\ &= \lim_{n \rightarrow \infty} (\langle Y_0, \Phi_n(0) \rangle_{\mathbb{L}^2} + \langle f, \varphi_n \rangle_{L^2(\Omega_T)} + \langle g, \varphi_{n, \Gamma} \rangle_{L^2(\Gamma_T)}) \\ &\leq \limsup_{n \rightarrow \infty} (\|\Phi_n(0)\|_{\mathbb{L}^2} + \|e^{-s\tilde{\alpha}} \tilde{\xi}^{3/2} \Phi_n\|_{L^2(0, T; \mathbb{L}^2)}) (\|Y_0\|_{\mathbb{L}^2} + \|f\|_{\tilde{Z}_\Omega} + \|g\|_{\tilde{Z}_\Gamma}). \end{aligned} \quad (4.12)$$

The backward version of estimate (2.20) on  $[0, t]$  yields

$$\|\Phi_n(\tau)\|_{\mathbb{L}^2}^2 \leq C (\|\Phi_n(t)\|_{\mathbb{L}^2}^2 + \|L^* \Phi_n\|_{L^2(\tau, t; \mathbb{L}^2)}^2) \quad (4.13)$$

for all  $0 \leq \tau \leq t \leq T$ , where  $C$  is uniform in  $a$  and  $b$ . We take  $\tau = 0$  and integrate over  $t \in (T/4, 3T/4)$  in (4.13). The Carleman estimate then implies

$$\begin{aligned} \|\Phi_n(0)\|_{\mathbb{L}^2}^2 &\leq C \int_{T/4}^{3T/4} \|\Phi_n(t)\|_{\mathbb{L}^2}^2 dt + C \|L^* \Phi_n\|_{L^2(0, 3T/4; \mathbb{L}^2)}^2 \\ &\leq C \|e^{-s\alpha} \xi^{3/2} \Phi_n\|_{L^2(T/4, 3T/4; \mathbb{L}^2)}^2 + C \|e^{-s\tilde{\alpha}} L^* \Phi_n\|_{L^2(0, 3T/4; \mathbb{L}^2)}^2 \\ &\leq C \|e^{-s\alpha} \xi^{3/2} \Phi_n\|_{L^2(0, T; \mathbb{L}^2)}^2 + C \|e^{-s\tilde{\alpha}} L^* \Phi_n\|_{L^2(0, T; \mathbb{L}^2)}^2 \\ &\leq C \|e^{-s\alpha} \xi^{3/2} \varphi_n\|_{L^2(\omega_T)}^2 + C \|e^{-s\alpha} L^* \Phi_n\|_{L^2(0, T; \mathbb{L}^2)}^2 + C \|e^{-s\tilde{\alpha}} L^* \Phi_n\|_{L^2(0, T; \mathbb{L}^2)}^2 \\ &\leq C \int_{\omega_T} |\varphi_n|^2 dx dt + C \|e^{-s\tilde{\alpha}} L^* \Phi_n\|_{L^2(0, T; \mathbb{L}^2)}^2, \end{aligned}$$

$$\limsup_{n \rightarrow \infty} \|\Phi_n(0)\|_{\mathbb{L}^2}^2 \leq C \|v_\varepsilon\|_{L^2(\omega_T)}^2 + C \|\rho_\varepsilon Y_\varepsilon\|_{L^2(0, T; \mathbb{L}^2)}^2,$$

using also that  $e^{-s\tilde{\alpha}} \rho_\varepsilon \leq 1$ . If we let  $\tau \in (0, T/4)$  and  $t = \tau + T/4$  in (4.13), we further derive

$$\begin{aligned} \|e^{-s\tilde{\alpha}} \tilde{\xi}^{3/2} \Phi_n\|_{L^2(0, T/4; \mathbb{L}^2)}^2 &\leq C \|\Phi_n\|_{L^2(0, T/4; \mathbb{L}^2)}^2 \\ &\leq C \|\Phi_n\|_{L^2(T/4, T/2; \mathbb{L}^2)}^2 + C \|L^* \Phi_n\|_{L^2(0, T/2; \mathbb{L}^2)}^2 \\ &\leq C \|e^{-s\alpha} \xi^{3/2} \Phi_n\|_{L^2(T/4, T/2; \mathbb{L}^2)}^2 + C \|e^{-s\tilde{\alpha}} L^* \Phi_n\|_{L^2(0, T/2; \mathbb{L}^2)}^2, \\ \|e^{-s\tilde{\alpha}} \tilde{\xi}^{3/2} \Phi_n\|_{L^2(0, T; \mathbb{L}^2)}^2 &\leq C \|e^{-s\alpha} \xi^{3/2} \Phi_n\|_{L^2(0, T; \mathbb{L}^2)}^2 + C \|e^{-s\tilde{\alpha}} L^* \Phi_n\|_{L^2(0, T; \mathbb{L}^2)}^2. \end{aligned}$$

As above, the Carleman estimate now yields

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|e^{-s\tilde{\alpha}} \tilde{\xi}^{3/2} \Phi_n\|_{L^2(0,T;\mathbb{L}^2)}^2 &\leq C \limsup_{n \rightarrow \infty} (\|\varphi_n\|_{L^2(\omega_T)}^2 + \|e^{-s\tilde{\alpha}} L^* \Phi_n\|_{L^2(0,T;\mathbb{L}^2)}^2) \\ &\leq C \|v_\varepsilon\|_{L^2(\omega_T)}^2 + C \|\rho_\varepsilon Y_\varepsilon\|_{L^2(0,T;\mathbb{L}^2)}^2. \end{aligned}$$

Dividing (4.12) by  $\|\rho_\varepsilon Y_\varepsilon\|_{L^2(0,T;\mathbb{L}^2)} + \|v_\varepsilon\|_{L^2(\omega_T)}$ , we arrive at

$$\|Y_\varepsilon\|_{L^2(0,T;\mathbb{L}^2)} \leq \|\rho_\varepsilon Y_\varepsilon\|_{L^2(0,T;\mathbb{L}^2)} + \|v_\varepsilon\|_{L^2(\omega_T)} \leq C(\|Y_0\|_{\mathbb{L}^2} + \|f\|_{\tilde{Z}_\Omega} + \|g\|_{\tilde{Z}_\Gamma}), \quad (4.14)$$

where the constant  $C$  is uniform in  $a$ ,  $b$ , and  $\varepsilon$ .

*Step 3.* As a consequence of (4.14) we find  $(y_*, v_*) \in \tilde{X} \times L^2(\omega_T)$  such that, up to subsequences,  $(Y_\varepsilon, v_\varepsilon)$  tends to  $(Y_*, v_*)$  weakly in  $L^2(0, T; \mathbb{L}^2) \times L^2(\omega_T)$  and  $\rho_\varepsilon Y_\varepsilon$  converge to some  $Z$  weakly in  $L^2(0, T; \mathbb{L}^2)$  as  $\varepsilon \rightarrow 0$ . Note that  $\rho_\varepsilon$  tends pointwise to  $e^{s\tilde{\alpha}}$  and  $0 \leq \rho_\varepsilon \leq e^{s\tilde{\alpha}}$ . If  $\psi \in L^2(0, T; \mathbb{L}^2)$  is supported in  $(0, T)$ , the functions  $\rho_\varepsilon \psi$  thus converge to  $e^{s\tilde{\alpha}} \psi$  in  $L^2(0, T; \mathbb{L}^2)$  by dominated convergence. It follows that  $Z = e^{s\tilde{\alpha}} Y_*$  and hence  $Y_* \in \tilde{X}$ . The limit  $(Y_*, v_*) \in \tilde{X} \times L^2(\omega_T)$  is still a distributional solution of (4.1)–(4.4) with  $Y_*(T, \cdot) = 0$ , i.e.,  $(Y_*, v_*) \in M$ .

Therefore,  $J$  has a unique minimizer  $(Y, v) \in M$  on the nonempty subset  $M$ . Using the weak convergence and (4.14), we derive the estimate (4.11) for  $(Y, v)$  by

$$\begin{aligned} \|Y\|_{\tilde{X}} + \|v\|_{L^2(\omega_T)} &\leq C J(Y, v)^{1/2} \leq C J(Y_*, v_*)^{1/2} \leq C(\|e^{s\tilde{\alpha}} Y_*\|_{L^2(0,T;\mathbb{L}^2)} + \|v_*\|_{L^2(\omega_T)}) \\ &\leq C \liminf_{\varepsilon \rightarrow 0} (\|\rho_\varepsilon Y_\varepsilon\|_{L^2(0,T;\mathbb{L}^2)} + \|v_\varepsilon\|_{L^2(\omega_T)}) \\ &\leq C(\|Y_0\|_{\mathbb{L}^2} + \|f\|_{\tilde{Z}_\Omega} + \|g\|_{\tilde{Z}_\Gamma}). \end{aligned} \quad (4.15)$$

*Step 4.* Finally, let  $f \in \tilde{Z}_\Omega$  and  $g \in \tilde{Z}_\Gamma$  be the given inhomogeneities. Consider the solution set  $M$  with respect to  $f$  and  $g$ . Choose  $f_n \in \tilde{Z}_\Omega$  and  $g_n \in \tilde{Z}_\Gamma$  with compact support in  $(0, T]$  such that  $f_n \rightarrow f$  in  $\tilde{Z}_\Omega$  and  $g_n \rightarrow g$  in  $\tilde{Z}_\Gamma$  as  $n \rightarrow \infty$ . Let  $(Y_n, v_n)$  be the corresponding minimizers of  $J$  obtained in Step 3 (where  $M = M_n$  is defined for  $f_n$  and  $g_n$ ). Since these functions satisfy (4.11), we find a subsequence such that  $(Y_{n_j}, v_{n_j})$  tends to some  $(Y_*, v_*)$  weakly in  $\tilde{X} \times L^2(\omega_T)$  as  $j \rightarrow \infty$ . The limit  $(Y_*, v_*)$  is a distributional solution of (4.1)–(4.4) with  $Y_*(T, \cdot) = 0$  so that  $(Y_*, v_*) \in M$ . This implies as before that  $J$  has a unique minimizer  $(Y, v)$  on  $M$ . The estimate (4.11) for  $(Y, v)$  can be shown as in (4.15).  $\square$

**4.2. Interior control for semilinear equations.** We now study the inhomogeneous semilinear problem

$$\partial_t y - d\Delta y + F(y) = v \mathbb{1}_\omega + f(t, x) \quad \text{in } \Omega_T, \quad (4.16)$$

$$\partial_t y_\Gamma - \delta \Delta_\Gamma y_\Gamma + d \partial_\nu y + G(y_\Gamma) = g(t, x) \quad \text{on } \Gamma_T, \quad (4.17)$$

$$y_\Gamma(t, x) = y|_\Gamma(t, x) \quad \text{on } \Gamma_T, \quad (4.18)$$

$$(y(0, \cdot), y_\Gamma(0, \cdot)) = (y_0, y_{0,\Gamma}) \quad \text{in } \Omega \times \Gamma. \quad (4.19)$$

**Theorem 4.4.** *Assume  $F, G \in C^1(\mathbb{R})$  satisfy*

$$F(0) = G(0) = 0 \quad \text{and} \quad |F(\xi)| + |G(\xi)| \leq C(1 + |\xi|) \quad \text{for } \xi \in \mathbb{R}.$$

*Then for all data  $Y_0 := (y_0, y_{0,\Gamma}) \in \mathbb{H}^1$ ,  $f \in \tilde{Z}_\Omega$  and  $g \in \tilde{Z}_\Gamma$  there is a control  $v \in L^2(\omega_T)$  such that (4.16)–(4.19) has a unique strong solution  $(y, y_\Gamma) \in \mathbb{E}_1 \cap \tilde{X}$  with  $y(T, \cdot) = y_\Gamma(T, \cdot) = 0$ .*

*Proof.* The result is proved by Schauder's fixed point theorem. Based on Proposition 4.3, we can follow the arguments given in Theorem 3.1 of [19].

We write  $F(\xi) = \tilde{F}(\xi)\xi$  and  $G(\xi) = \tilde{G}(\xi)\xi$  for bounded functions  $\tilde{F}$  and  $\tilde{G}$ . Fix data  $Y_0 := (y_0, y_{0,\Gamma}) \in \mathbb{H}^1$ ,  $f \in \tilde{Z}_\Omega$  and  $g \in \tilde{Z}_\Gamma$ . Take  $Y := (y, y_\Gamma) \in L^2(0, T; \mathbb{L}^2)$ . Proposition 4.3 then yields a unique minimizer  $(Z_*, v_*) =: \Phi(Y)$  of  $J$  among all distributional solutions in  $\tilde{X} \times L^2(\omega_T)$  of the linearized system

$$\partial_t z - d\Delta z + \tilde{F}(y(t, x))z = v \mathbb{1}_\omega + f(t, x) \quad \text{in } \Omega_T,$$

$$\partial_t z_\Gamma - \delta \Delta_\Gamma z_\Gamma + d \partial_\nu z + \tilde{G}(y_\Gamma(t, x))z = g(t, x) \quad \text{on } \Gamma_T, \quad (4.20)$$

$$y_\Gamma(t, x) = y|_\Gamma(t, x) \quad \text{on } \Gamma_T, \quad (4.21)$$

$$z(T, \cdot) = z_\Gamma(T, \cdot) = 0, \quad (z(0, \cdot), z_\Gamma(0, \cdot)) = Y_0 \quad \text{in } \bar{\Omega}.$$

We have thus defined a map  $\Phi : L^2(0, T; \mathbb{L}^2) \rightarrow \tilde{X} \times L^2(\omega_T)$ . We write  $Z_* = \Phi_1(Y)$  for the first component of  $\Phi(Y)$ . We consider  $\Phi_1$  as an operator  $\Phi_1 : L^2(0, T; \mathbb{L}^2) \rightarrow L^2(0, T; \mathbb{L}^2)$ .

Since  $\tilde{F}, \tilde{G}$  are bounded, the estimate (4.11) in Proposition 4.3 shows that  $\Phi_1$  maps all  $Y \in L^2(0, T; \mathbb{L}^2)$  into a ball in  $L^2(0, T; \mathbb{L}^2)$ . Moreover, Proposition 2.5 says that  $\Phi_1(Y)$  is even a strong solution of (4.20). Proposition 2.4 and (4.11) thus yield

$$\begin{aligned} \|\Phi_1(Y)\|_{\mathbb{E}_1} &\leq C(\|Y_0\|_{\mathbb{H}^1} + \|v\|_{L^2(\omega_T)} + \|f\|_{L^2(\Omega_T)} + \|g\|_{L^2(\Gamma_T)}) \\ &\leq C(\|Y_0\|_{\mathbb{H}^1} + \|f\|_{\tilde{Z}_\Omega} + \|g\|_{\tilde{Z}_\Gamma}), \end{aligned}$$

where  $C$  does not depend on  $Y$ . Since  $\mathbb{E}_1$  is compactly embedded into  $L^2(0, T; \mathbb{L}^2)$  by Proposition 2.2, we conclude that  $\Phi_1$  is compact.

Employing Proposition 4.3, one can prove that  $\Phi_1$  is continuous as in Theorem 3.1 of [19] by a contradiction argument. Schauder's fixed point theorem thus gives a function  $\hat{Y} \in L^2(0, T; \mathbb{L}^2)$  satisfying  $\hat{Y} = \Phi_1(\hat{Y}) \in \mathbb{E}_1$ . Hence, there is a control  $\hat{v} \in L^2(\omega_T)$  such that (4.20) holds with  $Z = \hat{Y} \in \tilde{X}$  and  $v = \hat{v}$ , as asserted.  $\square$

**4.3. Boundary control.** We look at the boundary control problem

$$\partial_t y - d\Delta y + F(y) = 0 \quad \text{in } \Omega_T, \quad (4.22)$$

$$\partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d\partial_\nu y + G(y_\Gamma) = \mathbb{1}_{\Gamma_0} v(t, x) \quad \text{on } \Gamma_T, \quad (4.23)$$

$$y_\Gamma(t, x) = y|_\Gamma(t, x) \quad \text{on } \Gamma_T, \quad (4.24)$$

$$(y(0, \cdot), y_\Gamma(0, \cdot)) = (y_0, y_{0,\Gamma}) \quad \text{in } \Omega \times \Gamma \quad (4.25)$$

where  $\Gamma_0 \Subset \Gamma$  is relatively open and non-empty. For simplicity we do not include inhomogeneities and potential terms. (One would need more regularity as above for them.) As in [4, 13] one can obtain null controllability at the boundary by means of an interior control on an enlarged domain  $\tilde{\Omega}$ . Dynamical boundary conditions require however more regularity than Dirichlet or Neumann control. We deal with this problem employing local regularity results, but cannot obtain quite the same level of regularity as before. We use the standard  $L^p$ -based fractional Sobolev spaces in this result and take more regular initial functions.

**Theorem 4.5.** *Let  $F$  and  $G$  satisfy the conditions of Theorem 4.4 and  $Y_0 := (y_0, y_{0,\Gamma}) \in \mathbb{H}^2$  with  $Y_0 \in W_p^{2-2/p}(\Omega)$  for some  $p > (N+2)/2$ . Then there is a control  $v \in L^2((0, T); L_{\text{loc}}^2(\Gamma_1))$  for a relatively open subset  $\Gamma_1 \subset \Gamma_0$  such that the problem (4.22)–(4.25) has a solution  $(y, y_\Gamma)$  satisfying  $y(T, \cdot) = y_\Gamma(T, \cdot) = 0$ . (We extend  $v$  by 0 to  $\Gamma_0$ .) This solution is contained in  $H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$  and has a trace in  $H^1(0, T; L^2(\Gamma')) \cap L^2(0, T; H^2(\Gamma'))$  for  $\Gamma' = (\Gamma \setminus \bar{\Gamma}_1) \cup \Gamma''$  and any  $\Gamma'' \Subset \Gamma_1$ .*

*Proof.* As in [4, 13], we find a larger domain  $\tilde{\Omega} \supset \Omega$  with  $\partial\tilde{\Omega} \in C^2$  such that  $\Gamma_1 := \partial\Omega \setminus \partial\tilde{\Omega} \subset \Gamma_0$  and the boundary of  $\Gamma_1$  within  $\Gamma_0$  has surface measure 0. Fix a function  $Z_0 \in \mathbb{H}^2(\tilde{\Omega})$  such that  $Z_0 \in W_p^{2-2/p}(\tilde{\Omega})$  and  $Z_0 = Y_0$  on  $\Omega$ . Take any domain  $\omega \Subset \tilde{\Omega} \setminus \bar{\Omega}$ . The interior null controllability result Theorem 4.4 gives a control  $u \in L^2(\omega_T)$  such that the solution  $(z, z_\Gamma) \in \mathbb{E}_1(\tilde{\Omega})$  of

$$\begin{aligned} \partial_t z - d\Delta z + F(z) &= \mathbb{1}_\omega u(t, x), \quad t \geq 0, \quad x \in \tilde{\Omega}, \\ \partial_t z_\Gamma - \delta\Delta_\Gamma z_\Gamma + d\partial_\nu z + G(z_\Gamma) &= 0, \quad t \geq 0, \quad x \in \partial\tilde{\Omega}, \\ z_\Gamma(t, x) &= z|_\Gamma(t, x), \quad t \geq 0, \quad x \in \partial\tilde{\Omega}, \\ (z(0, \cdot), z_\Gamma(0, \cdot)) &= Z_0, \quad x \in \Omega, \end{aligned} \quad (4.26)$$

satisfies  $z(T, \cdot) = z_\Gamma(T, \cdot) = 0$  on the closure of  $\tilde{\Omega}$ . Take an open set  $V \Subset \tilde{\Omega} \setminus \bar{\omega}$  which intersects  $\Gamma_1$  and has a smooth boundary. (Note that we cannot obtain  $\Gamma_1 \subset V$  here.) On  $V$  the function  $z$  satisfies the semilinear heat equation

$$\begin{aligned} \partial_t z - d\Delta z + F(z) &= 0, \quad 0 \leq t \leq T, \quad x \in V, \\ z(0, \cdot) &= z_0, \quad x \in V. \end{aligned} \quad (4.27)$$

By Proposition 2.2 and Sobolev's embedding, the restriction of  $z$  belongs to  $C([0, T]; H^1(V))$  and hence in  $C([0, T]; L^{p_1}(V))$  for  $p_1 = 2N/(N - 2)$  for  $N \geq 3$  and for all  $p_1 \in [1, \infty)$  if  $N = 2$ . Therefore the nonlinearity  $F(z)$  also belongs to  $C([0, T]; L^{p_1}(V))$ . Take any  $V_1 \Subset V$ . From local parabolic regularity theory we deduce that  $z \in L^{p_1}(0, T; W_{p_1}^2(V_1)) \cap W_{p_1}^1(0, T; L^{p_1}(V_1))$  and hence  $z \in C([0, T]; W_{p_1}^{2-2/p_1}(V_1))$ . (See estimate (IV.10.12) and Lemma II.3.4 in [23].) Sobolev's embedding then yields that  $z$  is contained in  $C([0, T]; L^{p_2}(V_1))$  with

$$p_2 = \frac{Np_1}{N + 2 - 2p_1} \quad \text{if } p_1 < \frac{N + 2}{2},$$

$p_2 = \infty$  if  $p_1 > (N + 2)/2$  and any  $p_2 < \infty$  if  $p_1 = (N + 2)/2$ . We can iterate this procedure, and after finitely many steps we obtain that  $z$  is bounded on  $[0, T] \times V'$  for some  $V' \Subset V$  which is arbitrarily close to  $V$  and has a smooth boundary.

In a next step we differentiate (4.27) with respect to  $x_j$  and obtain

$$\begin{aligned} \partial_t \partial_j z - d\Delta \partial_j z &= -F'(z) \partial_j z, & 0 \leq t \leq T, \quad x \in V', \\ \nabla z(0, \cdot) &= \nabla z_0, & x \in V', \end{aligned} \quad (4.28)$$

where the left hand side of the differential equation is understood in  $H^{-1}(V')$ , at first. Take  $V'' \Subset V'$  with a smooth boundary and a function  $\chi \in C_c^\infty(V')$  which is equal to 1 on  $V''$ . Then  $\chi \partial_j z$  satisfies (4.28) with zero Dirichlet boundary conditions and an additional inhomogeneity in  $L^2((0, T) \times V')$ . This problem has a solution  $w \in H^1(0, T; L^2(V')) \cap L^2(0, T; H^2(V') \cap H_0^1(V'))$  by standard parabolic theory. As in Proposition 2.5 one sees that  $w = \chi \partial_j z$  and hence  $z$  is contained in  $H^1(0, T; H^1(V'')) \cap L^2(0, T; H^3(V''))$ . In particular, the trace of  $z$  on  $V'' \cap \Gamma_1$  is an element of  $H^1(0, T; H^{1/2}(V'' \cap \Gamma_1)) \cap L^2(0, T; H^{5/2}(V'' \cap \Gamma_1))$ . Varying  $V$ , we infer that the trace  $z|_{\Gamma_1}$  exists in  $H^1(0, T; H_{\text{loc}}^{1/2}(\Gamma_1)) \cap L^2(0, T; H_{\text{loc}}^{5/2}(\Gamma_1))$ . We then define the control

$$v := (\partial_t z|_{\Gamma_1} - \delta \Delta_\Gamma z|_{\Gamma_1} + d\partial \nu z|_{\Gamma_1} + G(z|_{\Gamma_1}))$$

in  $L^2(0, T; L_{\text{loc}}^2(\Gamma_1))$  and extend it by 0 to  $\Gamma_0$ . Observe that the trace of  $z$  on  $\Gamma_2 := \Gamma \setminus \overline{\Gamma_1}$  belongs to  $H^1(0, T; L^2(\Gamma_2)) \cap L^2(0, T; H^2(\Gamma_2))$ . Let  $y$  be the restriction of  $z$  to  $\Omega$ . Then  $y$  is contained in  $H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$  and has a trace in  $H^1(0, T; L^2(\Gamma')) \cap L^2(0, T; H^2(\Gamma'))$  where  $\Gamma' = \Gamma_2 \cup \Gamma''$  for any  $\Gamma'' \Subset \Gamma_1$ . By construction  $y$  satisfies (4.22)–(4.25) and  $y(T, \cdot) = y_\Gamma(T, \cdot) = 0$  on  $\Omega \times \Gamma$ .  $\square$

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