Gradient systems

The 13th International Internet Seminar on Evolution Equations is intended to be an introduction to the analysis of evolution equations with gradient structure, with an emphasis on nonlinear parabolic equations. Gradient systems arise naturally in many physical models such as diffusion models, phase separation models and in surface evolution, but they also arise artificially as steepest descent methods in root finding algorithms, in various optimisation problems or in image analysis. The dissipation of some physical or abstract energy has an influence on the wellposedness of these gradient systems and on the qualitative behaviour of their solutions, including parabolic regularity, order preserving properties, or long-time behaviour. In this introductory course, starting from finite dimensional models, we will discuss some of these aspects. The lectures are at a beginning graduate level and assume only basic familiarity with Functional Analysis.

Organised by the European consortium "International School on Evolution Equations”, the annual Internet Seminars introduce graduate students, PhD students and postdocs to varying subjects related to Evolution Equations. The course consists of three phases. In Phase 1 (October-February), a weekly lecture is provided via the ISEM website. In Phase 2 (March-May), the participants form small international groups to work on projects. The projects complement the theory of Phase 1 and provide some applications of it. Phase 3 (13-19 June 2010) consists of the final one-week workshop in Sporthotel Kácov (Czech Republic). The teams present their projects, and lectures will be delivered by leading experts in the field.

ISEM team:

Virtual lecturers: RALPH CHILL (Metz)  
EVA FAŠANGOVÁ (Prague/Ulm)

Organization committee: TOMÁŠ BÁRTA (Prague)  
RALPH CHILL (Metz)  
EVA FAŠANGOVÁ (Prague/Ulm)

Website: http://isem.univ-metz.fr/

Contact/Registration: isem@mff.cuni.cz
Description of the course

An abstract gradient system is a differential equation of the form
\[ \dot{u} + \nabla E(u) = 0, \]
where $E$ is a real-valued function on a Banach space and $\nabla E$ is the gradient with respect to some ambient inner product or metric giving the Banach space a geometric structure. It is remarkable that gradient systems arise both naturally in physical, biological or chemical evolution problems such as diffusion problems, phase separation problems, geometric flows, ... and artificially as solution methods of various mathematical problems such as algebraic equations, optimisation problems, or problems from image analysis. Already in 1847, Augustin Cauchy proposed to use a steepest descent algorithm in order to solve a nonlinear equation, and the continuous version of Newton’s method from 1679 is also a gradient system.

Gradient systems have the particular feature that the underlying function $E$ is decreasing along solutions, and if $E$ is constant along a solution then the solution is already constant; gradient systems are for this reason the prototype examples of dissipative evolution equations\(^1\). The function $E$ is very often a physical energy or other characteristic quantity which dissipates.

Topics to be covered in this course include
- Gradient systems in finite dimensional spaces,
- Gradient systems in infinite dimensional spaces (well-posedness),
- The Laplace operator and the $p$-Laplace operator,
- Linear and nonlinear diffusion equations,
- Qualitative behaviour of solutions of gradient systems, including asymptotic behaviour,
- More examples of gradient systems (mainly parabolic equations).

Our aim is to give a thorough introduction to the field, at a speed suitable for master or beginning Ph.D students. Familiarity with basic topology, measure theory and elementary functional analysis is assumed, but will also be recalled during the course in separate appendices. There will be altogether 14 lectures. Each lecture will be accompanied by several exercises, the solutions to which shall be provided by the participants.

\(^1\) to dissipate (lat.: dissipare): to cause to lose energy (such as heat) irreversibly (http://www.thefreedictionary.com/). Other explanations are: to spend or expend intemperately or wastefully. Or: to attenuate to or almost to the point of disappearing.