Chapter 2

Finite element methods for elliptic boundary value problems

2.1 Lax-Milgram lemma

2.2 Sobolev spaces

In this section, we compile a number of important definitions and results from analysis without proofs.

Goal: Construct completions of “classical” function spaces (such as $C^1(\Omega) \cap C(\bar{\Omega})$).

Proposition 2.2.1 (Completions) If $V = (V, \langle \cdot, \cdot \rangle)$ is a vector space with inner product $\langle \cdot, \cdot \rangle$, then there is a unique (up to isometric isomorphisms) Hilbert space $(\overline{V}, \langle \cdot, \cdot \rangle)$ such that $V$ is dense in $\overline{V}$ and

$$\langle \langle u, v \rangle \rangle = \langle u, v \rangle \quad \text{for all } u, v \in V \subseteq \overline{V}.$$ 

The space $(\overline{V}, \langle \cdot, \cdot \rangle)$ is called the completion of $(V, \langle \cdot, \cdot \rangle)$. The completion of $V$ is the smallest complete space which contains $V$.

Idea of the proof: Consider equivalence classes of Cauchy sequences in $V$.

$$(v_n)_n \sim (u_n)_n \iff \lim_{n \to \infty} \|v_n - u_n\| = 0$$

Similar to the completion $\mathbb{Q} \rightarrow \mathbb{R}$.

Assumption for the rest of this chapter: $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain with boundary $\Gamma$. This means:

- $\Omega$ is open, bounded, connected, and non-empty.
• For every boundary point \( x \in \Gamma \), there is a neighbourhood \( U \) such that, after an affine change of coordinates (translation and/or rotation), \( \Gamma \cap U \) is described by the equation \( x_d = \chi(x_1, \ldots, x_{d-1}) \) with a uniformly Lipschitz continuous function \( \chi \). Moreover, \( \Omega \cap U \) is on one side of \( \Gamma \cap U \).

**Example:** Circles and rectangles are Lipschitz domains, but a circle with a cut is not.

**Definition 2.2.2 \((L_p(\Omega)\) spaces)**

For \( p \in \mathbb{N} \):

\[
L_p(\Omega) := \left\{ v : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |v(x)|^p \, dx < \infty \right\}
\]

\[
\|v\|_{L_p(\Omega)} := \left( \int_\Omega |v(x)|^p \, dx \right)^{\frac{1}{p}}
\]

For \( p = \infty \):

\[
L_\infty(\Omega) := \{ v : \Omega \to \mathbb{R} \text{ measurable and } |v(x)| < \infty \text{ a.e.} \}
\]

\[
\|v\|_{L_\infty(\Omega)} := \inf \{ c > 0 : |v(x)| \leq c \text{ a.e.} \}
\]

The integral is the Lebesgue integral, and the abbreviation “a.e.” means “almost everywhere”, i.e. for all \( x \in \Omega \setminus N \) for null sets \( N \).

For convenience, we write \( \|v\|_{L_p} \) instead of \( \|v\|_{L_p(\Omega)} \) if it is clear which domain is meant.

The elements of \( L_p(\Omega) \) are not functions but equivalence classes of functions: \( u, v \in L_p(\Omega) \) are equivalent if \( u = v \) a.e.. It is common usage to speak of “\( L_p \) functions”, but some care is required: It does not make sense to speak about “the value of a \( L_p \) function in a point \( x \)” because a single point is a null set.

**Important properties:**

- \( \| \cdot \|_{L_p} \) is a norm on \( L_p(\Omega) \) for every \( p \in \{1, 2, \ldots, \infty \} \), and \((L_p(\Omega), \| \cdot \|_{L_p})\) is a Banach space (complete).

- Special case \( p = 2 \): The space \( L_2(\Omega) \) is a Hilbert space with inner product

\[
\langle u, v \rangle_{L_2(\Omega)} = \int_\Omega u(x)v(x) \, dx.
\]

**Definition 2.2.3**

\[
L_{1}^{loc}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ is measurable and locally integrable} \right\}
\]

“Locally integrable” means that \( u \in L^1(S) \) for all compact subsets \( S \subset \Omega \).
Definition 2.2.4 (Weak derivative) A function \( u \in L^1_{\text{loc}}(\Omega) \) is weakly differentiable with respect to \( x_i \) if there is a \( v \in L^1_{\text{loc}}(\Omega) \) such that
\[
\int_{\Omega} v(x) \phi(x) \, dx = -\int_{\Omega} u(x) \partial_{x_i} \phi(x) \, dx \quad \text{for all } \phi \in C^\infty_c(\Omega)
\]
or equivalently
\[
\langle v, \phi \rangle_{L^2(\Omega)} = -\langle u, \partial_{x_i} \phi \rangle_{L^2(\Omega)} \quad \text{for all } \phi \in C^\infty_c(\Omega).
\]
Then, \( v \) is called the weak (partial) derivative and is denoted by \( v = \partial_{x_i} u \).

Uniqueness: If \( v = \partial_{x_i} u \in L^1_{\text{loc}}(\Omega) \) and \( \tilde{v} = \partial_{x_i} u \in L^1_{\text{loc}}(\Omega) \) are both weak partial derivatives of \( u \in L^1_{\text{loc}}(\Omega) \), then \( v = \tilde{v} \) a.e.

Consistency of the definition: If \( u \in C^1(\Omega) \cap C(\Omega) \), then the weak derivative coincides with the classical derivative (exercise).

Notation: For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \) we set
\[
\partial^\alpha u := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d} u, \quad |\alpha|_1 = \alpha_1 + \ldots + \alpha_d.
\]
The above definition can now be extended to higher-order weak derivatives: \( v(x) = \partial^\alpha u(x) \) is the weak partial derivative of \( u(x) \) if
\[
\int_{\Omega} v(x) \phi(x) \, dx = (-1)^{|\alpha|_1} \int_{\Omega} u(x) \partial^\alpha \phi(x) \, dx \quad \text{for all } \phi \in C^\infty_c(\Omega).
\]

Definition 2.2.5 (Sobolev space \( H^m(\Omega) \)) For \( m \in \mathbb{N} \) the Sobolev space \( H^m(\Omega) \) is the set
\[
H^m(\Omega) := \left\{ v \in L^2(\Omega) : \partial^\alpha v \in L^2(\Omega) \text{ exists for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha|_1 \leq m \right\}
\]
with inner product
\[
\langle u, v \rangle_{H^m(\Omega)} := \sum_{|\alpha|_1 \leq m} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)}.
\]

and norm
\[
\|v\|_{H^m(\Omega)} = \sqrt{\langle v, v \rangle_{H^m(\Omega)}}.
\]
As for the \( L^p \) spaces, we often write \( \langle u, v \rangle_{H^m} \) and \( \| \cdot \|_{H^m} \) instead of \( \langle u, v \rangle_{H^m(\Omega)} \) and \( \| \cdot \|_{H^m(\Omega)} \).

Example (\( m = 1 \)):
\[
H^1(\Omega) := \left\{ v \in L^2(\Omega) : \partial_{x_i} v \in L^2(\Omega) \text{ exists for all } i = 1, \ldots, d \right\}
\]
\[
\langle u, v \rangle_{H^1(\Omega)} := \langle u, v \rangle_{L^2(\Omega)} + \sum_{i=1}^d \langle \partial_{x_i} u, \partial_{x_i} v \rangle_{L^2(\Omega)}
\]
\[
\|v\|_{H^1(\Omega)} = \left( \|v\|^2_{L^2(\Omega)} + \sum_{i=1}^d \|\partial_{x_i} v\|^2_{L^2(\Omega)} \right)^{1/2}.
\]
Important properties:

\((H^m(\Omega), \langle \cdot, \cdot \rangle_{H^m})\) is a Hilbert space for every \(m \in \mathbb{N}_0\). (We define \(H^0(\Omega) = L^2(\Omega)\).) In fact, the space

\[
C^{\infty,m}(\Omega) := \left\{ v \in C^\infty(\Omega) : \int_\Omega |\partial^\alpha v(x)|^2 \, dx < \infty \text{ for all } \alpha \in \mathbb{N}_d^1 \text{ with } |\alpha|_1 \leq m \right\},
\]

is dense in \(H^m(\Omega)\) with respect to \(\| \cdot \|_{H^m}\): For every \(u \in H^m(\Omega)\) and every \(\varepsilon > 0\) there is a \(v_\varepsilon \in C^{\infty,m}(\Omega)\) such that \(\|v_\varepsilon - u\|_{H^m} < \varepsilon\).

It can be shown that \(H^m(\Omega)\) is the completion of \(C^{\infty,m}(\Omega)\) with respect to \(\| \cdot \|_{H^m}\), i.e.

\[
u \in H^m(\Omega) \iff \text{There are } v_n \in C^{\infty,m}(\Omega) \text{ such that } \lim_{n \to \infty} \| u - v_n \|_{H^m(\Omega)} = 0.
\]

Example: Let \(\Omega = (-1, 1)\) and \(u(x) = |x|\). Then \(u \in H^1(\Omega)\), but \(u \notin C^{\infty,1}(\Omega)\). However, we can define \(v_n := \sqrt{x^2 + \frac{1}{n^2}} \in C^{\infty,1}(\Omega)\) and show that \(\|v_n - u\|_{H^1(\Omega)} \to 0\) for \(n \to \infty\) (exercise).

**Definition 2.2.6** The **Sobolev space** \(H^m_0(\Omega)\) is the completion of \(C^\infty_c(\Omega)\) with respect to \(\| \cdot \|_{H^m(\Omega)}\), i.e.

\[
u \in H^m_0(\Omega) \iff \text{There are } v_n \in C^\infty_c(\Omega) \text{ such that } \lim_{n \to \infty} \| u - v_n \|_{H^m(\Omega)} = 0.
\]

\(H^m_0(\Omega)\) is a closed subspace of \(H^m(\Omega)\). If the boundary \(\Gamma\) is \(C^1\), then \(v \in C(\overline{\Omega}) \cap H^m_0(\Omega)\) implies that \(v(x) = 0\) for all \(x \in \Gamma\).

**Proposition 2.2.7** (Poincaré-Friedrichs inequality) Let

\[
|v|_{H^1(\Omega)} := \left( \sum_{i=1}^d \| \partial_{x_i} v \|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}
\]

denote the Sobolev seminorm. There is a constant \(c_\Omega > 0\) such that

\[
\|v\|_{H^1(\Omega)} \leq c_\Omega |v|_{H^1(\Omega)} \quad \text{for all } v \in H^1_0(\Omega).
\]

**Proof.** See 1.5, page 29 in [Bra07].

**Remark:** On \(H^1_0(\Omega)\), the seminorm \(| \cdot |_{H^1(\Omega)}\) is even a norm and \(| \cdot |_{H^1(\Omega)} \sim \| \cdot \|_{H^1(\Omega)}\) (exercise).

If we want to solve a PDE with inhomogeneous boundary conditions, we need a condition such as, e.g., \(u(x) = g(x)\) for all \(x \in \Gamma\). Since \(\Gamma\) is a null set, however, we have to specify how such a condition has to be understood if \(u \in L^2(\Omega)\).
Theorem 2.2.8 (Trace theorem) There is a bounded linear map
\[ \gamma : H^1(\Omega) \rightarrow L^2(\Gamma), \quad \|\gamma(v)\|_{L^2(\Gamma)} \leq c\|v\|_{H^1(\Omega)} \quad (c > 0) \]
such that \( \gamma(v) = v|_\Gamma \) for all \( v \in C^1(\bar{\Omega}) \)

Proof: page 45-47 in [Bra07]

Remark: It can be shown that
\[ H^1_0(\Omega) := \{ v \in H^1(\Omega) : \gamma(v) = 0 \} \]

Theorem 2.2.9 (Sobolev’s embedding theorem) Let \( \Omega \subseteq \mathbb{R}^d \) and let \( r, s \in \mathbb{N}_0 \) with \( r > s + d/2 \). Then
\[ H^r(\Omega) \subset C^s(\bar{\Omega}) \]
and there is a constant \( c > 0 \) such that
\[ \|u\|_{C^s(\bar{\Omega})} \leq c\|u\|_{H^r(\Omega)}, \quad \text{where} \quad \|u\|_{C^s(\bar{\Omega})} := \sum_{|\alpha| \leq s} \sup_{x \in \bar{\Omega}} |\partial^\alpha u(x)|. \]

Examples:
\begin{align*}
  d = 1 & \implies H^1(\Omega) \subset C(\bar{\Omega}), \quad H^2(\Omega) \subset C^1(\bar{\Omega}) \\
  d \in \{2, 3\} & \implies H^1(\Omega) \not\subset C(\bar{\Omega}), \quad H^2(\Omega) \subset C(\bar{\Omega})
\end{align*}

2.3 Weak solutions of elliptic boundary value problems

Let \( \Omega \subseteq \mathbb{R}^d \) be a bounded Lipschitz domain with \( d \in \{2, 3\} \) and assume that \( \Gamma \) is piecewise \( C^1 \). Consider the elliptic PDE
\[ Au = f \]
for \( f \in L^2(\Omega) \) and with the second-order differential operator
\[ Au(x) = -\text{div}(\kappa(x) \nabla u(x)) + \kappa_0(x)u(x) \]
\[ = -\sum_{i,j=1}^{d} \partial_{x_i}(\kappa_{ij}(x)\partial_{x_j} u(x)) + \kappa_0(x)u(x) \]
with \( \kappa(x) = (\kappa_{ij}(x))_{i,j} \in \mathbb{R}^{d \times d} \) for all \( x \in \Omega \). The coefficient functions \( \kappa_{ij} : \Omega \rightarrow \mathbb{R} \) and \( \kappa_0 : \Omega \rightarrow \mathbb{R} \) are supposed to have the following properties:

1. \( |\kappa_0(x)|, \ |\kappa_{ij}(x)| \leq M \quad \text{for all} \quad x \in \Omega \)
2. \( \kappa_{ij} = \kappa_{ji} \) for all \( i, j = 1, \ldots, d \)

3. There are constants \( c_0 \geq 0 \) and \( c_1 > 0 \) such that
\[
\sum_{i=1}^{d} \sum_{j=1}^{d} \kappa_{ij}(x)\xi_i\xi_j \geq c_1 \sum_{i=1}^{d} \xi_i^2
\]
for all \( x \in \Omega \) and \( \xi = (\xi_1, \ldots, \xi_d)^T \in \mathbb{R}^d \). (2.1)

The second assertion is equivalent to
\[
\xi^T \kappa \xi \geq c_1 \|\xi\|_2^2, \quad \xi = (\xi_1, \ldots, \xi_d)^T.
\]

A differential operator with the properties 2 and 3 is called **elliptic**.

**Example:** For \( \kappa_0(x) = 0 \), \( \kappa_{ii} = 1 \) and \( \kappa_{ij} = 0 \) for \( i \neq j \), we obtain \( A = -\Delta \).

1. **Homogeneous Dirichlet boundary conditions.** Consider the boundary value problem
\[
Au = f \quad \text{in } \Omega \\
u = 0 \quad \text{on } \Gamma.
\]

Assume that \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) is a classical solution. Multiply both sides of the first line with a test function \( v \in C^\infty_c(\Omega) \) and integrate over \( \Omega \):
\[
-\sum_{i,j=1}^{d} \int_{\Omega} \partial_{x_i} \left( \kappa_{ij}(x)\partial_{x_j} u(x) \right) v(x) \, dx + \int_{\Omega} \kappa_0(x)u(x)v(x) \, dx = \int_{\Omega} f(x)v(x) \, dx
\]

Apply Green’s formula:
\[
\sum_{i,j=1}^{d} \int_{\Omega} \kappa_{ij}(x)\partial_{x_j} u(x)\partial_{x_i} v(x) \, dx + \int_{\Omega} \kappa_0(x)u(x)v(x) \, dx = \int_{\Omega} f(x)v(x) \, dx =: \ell(v)
\]

Equivalent notation:
\[
a(u, v) = \int_{\Omega} (\nabla v(x))^T \kappa(x) \nabla u(x) \, dx + \int_{\Omega} \kappa_0(x)u(x)v(x) \, dx
\]

**Variational (weak) formulation:** Find \( u \in H^1_0(\Omega) \) such that
\[
a(u, v) = \ell(v) \quad \text{for all } v \in H^1_0(\Omega).
\]

Such an \( u \) is called a **weak solution** of the boundary value problem.

Since we seek a solution in \( H^1_0(\Omega) \), the boundary condition is fulfilled.

Show that the assumptions of the Lax-Milgram theorem (Theorem ??) are true:
• Show that $\ell : H^1_0(\Omega) \to \mathbb{R}$ is a continuous linear form:

$$|\ell(v)| = |\langle f, v \rangle|_{L^2} \leq \|f\|_{L^2} \|v\|_{L^2} \leq \|f\|_{L^2} \|v\|_{H^1}.$$  

• Show that $a(\cdot, \cdot)$ is a $H^1_0(\Omega)$-elliptic bilinear form. Boundedness:

$$|a(u,v)| \leq \sum_{i,j=1}^{d} \int_{\Omega} |\kappa_{ij}(x)| \cdot |\partial_{x_i} u(x)| \cdot |\partial_{x_j} v(x)| + \left( \sum_{i,j=1}^{d} \kappa_{ij}(x) \right) \cdot |u(x)| \cdot |v(x)| \, dx \leq \sum_{i,j=1}^{d} \langle |\partial_{x_j} u|, |\partial_{x_i} v| \rangle_{L^2} + M \langle |u|, |v| \rangle_{L^2} \leq M(d^2 + 1) \cdot \|u\|_{H^1} \|v\|_{H^1}.$$  

Coercivity:

$$a(v,v) = \int_{\Omega} \sum_{i,j=1}^{d} \kappa_{ij}(x) \partial_{x_i} v(x) \partial_{x_j} v(x) + \kappa_0(x)v^2(x) \, dx \geq \int_{\Omega} c_1 \sum_{i=1}^{d} (\partial_{x_i} v(x))^2 + c_0 v^2(x) \, dx \geq c_1 \|v\|^2_{H^1}.$$  

Applying Lax-Milgram (Theorem ??) with $V = H^1_0(\Omega)$ yields that the problem (2.2) has a unique (weak) solution $u \in H^1_0(\Omega)$.

**Remark:** The definition of the differential operator $\mathcal{A}$ only makes sense if $\kappa_{ij} \in C^1(\Omega)$, whereas only $\kappa_{ij} \in L^\infty(\Omega)$ is required in the weak formulation.

2. Inhomogeneous Dirichlet boundary conditions. For a given function $g : \Gamma \to \mathbb{R}$ we consider the problem

$$\mathcal{A}u = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \Gamma$$

Assumption: There is a $u_* \in H^1(\Omega)$ such that $g = u_*|_{\Gamma} = \gamma(u_*)$ in the sense of the trace theorem 2.2.8.
Variational (weak) formulation: Find \( u \in H^1(\Omega) \) (instead of \( H^1_0(\Omega) \)) such that \( w := u - u^* \in H^1_0(\Omega) \) and

\[
a(u, v) = \ell(v) \quad \text{for all } v \in H^1_0(\Omega)
\]

(same derivation as before). Equivalent: Find \( w \in H^1_0(\Omega) \) such that

\[
a(w, v) = \ell(v) - a(u^*, v) =: \tilde{\ell}(v) \quad \text{for all } v \in H^1_0(\Omega).
\]

Verify the conditions of the Lax-Milgram theorem: \( \tilde{\ell} : H^1_0(\Omega) \to \mathbb{R} \) is continuous, because

\[
|\tilde{\ell}(v)| \leq |\ell(v)| + |a(u^*, v)| = |(f, v)_{L^2}| \\
\leq \|f\|_{L^2} \|v\|_{H^1} + M(d^2 + 1)\|u^*\|_{H^1} \|v\|_{H^1} \\
= (\|f\|_{L^2} + M(d^2 + 1)\|u^*\|_{H^1}) \|v\|_{H^1}
\]

The Lax-Milgram theorem yields that (2.4) has a unique solution \( w \in H^1_0(\Omega) \). Hence, (2.3) has a unique solution \( u = u^* + w \in H^1(\Omega) \).

3. Neumann boundary conditions. For every \( x \in \Gamma \), we define the conormal derivative

\[
\frac{\partial u}{\partial \eta_\kappa}(x) := \sum_{i,j=1}^d \kappa_{ij}(x) \partial_{x_j} u(x) \eta_i(x) = \eta(x)^T \kappa(x) \nabla u(x),
\]

where \( \eta : \Gamma \to \mathbb{R}^d \) is again the outer unit normal. Let \( f \in L^2(\Omega) \) and \( g \in L^2(\Gamma) \) and consider the boundary value problem

\[
\begin{align*}
Au &= f & \text{in } \Omega \\
\frac{\partial u}{\partial \eta_\kappa} &= g & \text{on } \Gamma.
\end{align*}
\]

Variational (weak) formulation? Let \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) is a classical solution and proceed as in 1.: Multiply both sides of the first line with a test function \( v \in C^\infty(\Omega) \), integrate
over $\Omega$ and apply Green’s formula. This yields

$$\ell(v) = \int_{\Omega} f(x)v(x) \, dx = \int_{\Omega} Au(x)v(x) \, dx$$

$$= -\sum_{i,j=1}^{d} \int_{\Omega} \partial_{x_i} \left( \kappa_{ij}(x) \partial_{x_j} u(x) \right) v(x) \, dx + \int_{\Omega} \kappa_{0}(x) u(x)v(x) \, dx$$

$$= -\int_{\Gamma} \left( \sum_{i,j=1}^{d} \kappa_{ij}(x) \partial_{x_j} u(x) \eta_i(x) \right) v(x) d\sigma(x)$$

$$= -\int_{\Gamma} \kappa_{0}(x) u(x)v(x) \, dx$$

$$\quad + \int_{\Omega} \sum_{i,j=1}^{d} \kappa_{ij}(x) \partial_{x_j} u(x) \partial_{x_i} v(x) + \int_{\Omega} \kappa_{0}(x) u(x)v(x) \, dx$$

$$= -\int_{\Gamma} g(x)v(x)d\sigma(x) + a(u,v)$$

**Weak formulation:** Find $u \in H^1(\Omega)$ (instead of $H^1_0(\Omega)$) such that

$$a(u,v) = \ell(v) + \int_{\Gamma} g(x)v(x) \, d\sigma(x) =: \tilde{\ell}(v) \quad \text{for all } v \in H^1(\Omega) \quad (2.6)$$

with $a(\cdot, \cdot)$ as before. Lax-Milgram conditions:

- Show that $\ell : H^1(\Omega) \to \mathbb{R}$ is continuous: The trace theorem (Theorem 2.2.8) yields

$$|\tilde{\ell}(v)| \leq \|f\|_{L^2(\Omega)} \cdot \|v\|_{H^1(\Omega)} + \|g\|_{L^2(\Gamma)} \cdot \|\gamma(v)\|_{L^2(\Gamma)} \leq c\|v\|_{H^1(\Omega)}$$

$$\leq C\|v\|_{H^1(\Omega)}.$$

- Show that $a(\cdot, \cdot)$ is $H^1(\Omega)$-elliptic. Boundedness can be shown as before. To prove coercivity, it can be shown as in 1. that

$$a(v,v) \geq c_1\|v\|_{H^1}^2 + c_0\|v\|_{L^2}^2.$$

This time, the Poincaré-Friedrichs inequality cannot be applied. In order to obtain coercivity, we have to assume that $c_0 > 0$ (instead of $c_0 \geq 0$) and use that

$$c_1\|v\|_{H^1}^2 + c_0\|v\|_{L^2}^2 \geq \min\{c_1, c_0\} \cdot \|v\|_{H^1}^2.$$

Lax-Milgram $\rightarrow$ unique solution.

**Remark.** Let $\kappa_0(x) = 0$ for all $x \in \Omega$, i.e. $c_0 = 0$. In this case, the solution of the boundary value problem (2.5) is not unique (exercise). This indicates that the case $c_0 = 0$ may also cause problems in the original formulation of the problem.
Bibliography