Goal for the remainder of this chapter: Analyze the convergence of multigrid methods. In the next section, we develop an appropriate framework.

4. Analytical
4.1. Numerical Setting

4.1. Scales of norms

Let \( \langle \cdot, \cdot \rangle \) be the Euclidean scalar product on \( \mathbb{R}^N \) with norm \( \| \cdot \| \). Let \( A \in \mathbb{R}^{N \times N} \) be symmetric and positive definite with diagonalization

\[
A = Q \Lambda Q^T, \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}, \quad \lambda_i \text{ eigenvalues}
\]

\( Q \) orthogonal, \( \Lambda^T = \Lambda \), \( Q^T Q = QQ^T = I \)

For \( s \in \mathbb{R} \) and \( x \in \mathbb{R}^N \) we define

\[
A^S := Q \Lambda^S Q^T, \quad \Lambda^S = \begin{pmatrix} \lambda_1^S & & \\ & \ddots & \\ & & \lambda_N^S \end{pmatrix},
\]

\[
\| x \|_S := \sqrt{\langle x, x \rangle} = \sqrt{x^T A^S x} = \| A^{S/2} x \|_2
\]

It can be checked that \( \| \cdot \|_S \) is a norm on \( \mathbb{R}^N \).
Special cases: $\|x\|_0 = \|x\|$ Euclidean norm

$\|x\|_1 = \|Ax\|_1 = \sqrt{x^T Ax}$ "energy norm"

Lemma 4.1

For $s \leq r$, the norms $\|x\|_s$ have the following properties:

(a) Logarithmic convexity:

$$\|x\|_s \leq \|x\|_r^{s/r} \|x\|_t^{t/r}$$

for $s = \frac{r + t}{2}$

(b) Monotony: If $\lambda_{\min} > 0$ is the smallest eigenvalue of $A$ and $s \leq t$, then

$$\lambda_{\min} \|x\|_s \leq \lambda_{\min} \|x\|_t$$

(c) Shift theorem: If $Ax = b$, then

$$\|x\|_{s+2} = \|x\|_s$$

Proof: Exercise

### 4.2 Assumptions and notation

Consider a $\Omega$-elliptic boundary value problem on a bounded, convex domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary. Weak formulation:

Find $u \in V$ such that $a(u, v) = ll(v) \forall v \in V$

with $V = H^1_0(\Omega)$ or $V = H^1(\Omega)$. The problem is $H^2$-regular (cf. remark after Def. 6.6 in III)

Let $T_0$ be a coarse triangulation of $\Omega$. Construct refined triangulation $T_l$:

Divide every triangle into smaller copies by adding the three midpoints of the three edges as new vertices:

In the same way we construct $T_l$ from $T_{l-1}$ for $l = 2, \ldots, L \in \mathbb{N}$.

Every triangle is similar to a triangle in $T_0$. 
Let \( h_x = \max_{T \in \mathcal{T}_h} \text{diam}(T) \) be the maximal diameter in \( \mathcal{T}_h \).

By construction: \( h_x = \frac{1}{2} h_{x-1} \).

Consider linear finite elements:

Nodes \( z_j \); vertices \( q_j \) of triangles, \( q_j \) basis functions,

\( V^{(1)} \), space of piecewise linear functions with respect to \( \mathcal{T}_h \).

Warning: Throughout this chapter, until the end of this chapter, a different scaling of the basis functions will be used, namely:

\[
q_i^{(1)}(q_j^{(1)}) = \begin{cases} \frac{1}{h_x^2} & \text{if } i = j \\
0 & \text{else} \end{cases}
\]

Every \( v^{(1)} \in V^{(1)} \) has a representation

\[
v^{(1)} = \sum_{i=1}^{N} \hat{v}_i q_i^{(1)}, \quad \hat{v}_i = h_x v(q_i^{(1)}), \quad \hat{v}^{(1)} = (\hat{v}_i)_{i=1,...,N^{(1)}}
\]

and the mapping \( \hat{v}^{(1)} \rightarrow v^{(1)} \) is an isomorphism.

4.3 Grid-dependent norms

Let \( A^{(1)} \in \mathbb{R}^{N^{(1)} \times N^{(1)}} \) be the stiffness matrix corresponding to \( \mathcal{T}_h \):

\[
A^{(1)} = \left( a(q_i^{(1)}, q_j^{(1)}) \right)_{i,j}
\]

Define a mapping \( \alpha^{(1)} : V^{(1)} \rightarrow V^{(1)} \) via

\[
\alpha^{(1)} : v^{(1)} = \sum_{i=1}^{N} \hat{v}_i q_i^{(1)} \quad \mapsto \quad w^{(1)} = \sum_{i=1}^{N} \hat{w}_i q_i^{(1)} \quad \text{with} \quad \hat{w}_i = A^{(1)} \hat{v}_i
\]

For \( v^{(1)}, w^{(1)} \in V^{(1)} \) and \( w^{(1)} = \alpha^{(1)} v^{(1)} \) it follows that

\[
a(v^{(1)}, w^{(1)}) = \sum_{i,j} \hat{v}_i \hat{w}_j a(q_i^{(1)}, q_j^{(1)}) = (\hat{v}^{(1)})^T A^{(1)} \hat{v}^{(1)} = (\hat{w}^{(1)})^T A^{(1)} \hat{w}^{(1)} = \langle \hat{w}^{(1)}, \hat{v}^{(1)} \rangle
\]

Remark: \( \langle \cdot, \cdot \rangle \) Euclidean scalar product with norm \( \| \cdot \| \).

For \( v^{(1)} \in V^{(1)} \), \( \alpha^{(1)} v^{(1)} \in \mathbb{R}^{N^{(1)}} \), we define the scaled norms,

\[
\| v^{(1)} \|_s = \sqrt{\langle v^{(1)}, (A^{(1)})^{1/2} v^{(1)} \rangle} = \| (A^{(1)})^{1/2} v^{(1)} \|
\]

as in 4.1.
Lemma 4.1 (norm equivalence)

There are constants $c_0$, $C_0$ independent of $h$ and $h_2$ such that

$$c_0 \| \nu^{(v)} \|_{L_2(x_1)} \leq \| \nu^{(v)} \|_{L_2(x_1)} \leq C_0 \| \nu^{(v)} \|_{L_2(x_1)} \quad \forall v, u \in V^{(e)}.$$

Proof: Let $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ be the three vertices of $T \in \mathcal{T}_h$ (local enumeration), and let $m_{1}^T = \frac{1}{2} (\tilde{P}_1 + \tilde{P}_2)$, $m_{2}^T = \frac{1}{2} (\tilde{P}_2 + \tilde{P}_3)$, $m_{3}^T = \frac{1}{2} (\tilde{P}_3 + \tilde{P}_1)$ be the midpoints of the three edges of $T$. Let $|T|$ be the area of $T$.

Then, the quadrature formula

$$\sum_{T \in \mathcal{T}_h} \int \sum_{T \in \mathcal{T}_h} \frac{1}{3} \sum_{i=1}^{3} \frac{1}{2} \quad w(m_i^T) dx = \sum_{T \in \mathcal{T}_h} \frac{1}{3} \sum_{i=1}^{3} \frac{1}{2} \quad w(m_i^T)$$

is exact for polynomials of degree $\leq 2$ (exercise).

For all $\nu^{(e)} \in V^{(e)}$, it follows that

$$\| \nu^{(e)} \|_{L_2(x_1)}^2 = \sum_{T \in \mathcal{T}_h} \frac{1}{3} \sum_{i=1}^{3} \left( \nu^{(e)}(m_i^T) \right)^2 = \frac{1}{3} \sum_{T \in \mathcal{T}_h} \| \nu^{(e)}(m_i^T) \|^2$$

By construction of $\mathcal{T}_h$ there are constants $c^*, C^*$ such that

$$c^* h_2^2 \leq |T| \leq C^* h_2^2 \quad \forall T \in \mathcal{T}_h, \quad k = 1, \ldots, L.$$

Since $\nu^{(e)}$ is piecewise linear we have

$$\begin{pmatrix}
\nu^{(e)}(m_1^T) \\
\nu^{(e)}(m_2^T) \\
\nu^{(e)}(m_3^T)
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\nu^{(e)}(\tilde{P}_1) \\
\nu^{(e)}(\tilde{P}_2) \\
\nu^{(e)}(\tilde{P}_3)
\end{pmatrix}$$

with $C^{(v)} = h_2 \nu^{(e)}(\tilde{P}_1)$ this yields

$$\| \nu^{(e)} \|_{L_2(x_1)}^2 \leq C^* \frac{h_2^2}{3} \| M \|^2 \sum_{T \in \mathcal{T}_h} \left( \nu^{(e)}(\tilde{P}_1) + \nu^{(e)}(\tilde{P}_2) + \nu^{(e)}(\tilde{P}_3) \right)^2$$

$$\leq C \| M \|^2 \cdot C \sum \| \nu^{(v)} \|^2 \leq C \| M \|^2 \cdot \| \nu^{(v)} \|_0$$

where $C$ is the maximal number of triangles sharing a single vertex. The other bound follows analogously.