\[ \begin{align*}
\frac{\partial^2 u}{\partial t^2}(x,t) &= c^2 \frac{\partial^2 u}{\partial x^2}(x,t) + (-c)^2 \frac{\partial^2 u}{\partial x^2}(x,t) \\
\frac{\partial^2 u}{\partial x^2}(x,t) &= F''(x+ct) + G''(x-ct)
\end{align*} \]

Interpretation:
For \( V_0(x) = 0 \) the solution is simply the sum of two "copies" of the initial data \( u_0 \). These two copies travel to the left and right with speed \( c \) without changing their shape.

\[ u_0(x+c) \rightarrow u_0(x-ct) \]

The regularity of the solution does not improve.

(5) Solution on an interval

Consider now the wave equation on \( \Omega = (0, \pi) \) with homogeneous Dirichlet boundary conditions:

\[ \begin{align*}
\frac{\partial^2 u}{\partial t^2}(x,t) &= c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \\
x \in \Omega, \quad t \geq 0 \\
u(t,0) &= u(t,\pi) = 0 \\
u(0,x) &= u_0(x) \\
\frac{\partial u}{\partial t}(0,x) &= V_0(x)
\end{align*} \]

Assume that \( u_0 \in C^2(\Omega) \), \( V_0 \in C^0(\Omega) \) and that

\[ u_0(0) = u_0(\pi) = u_0''(0) = u_0''(\pi) = V_0(0) = V_0(\pi) = 0 \]
Define \(2\pi\)-periodic odd continuations of \(u_0\) and \(v_0\):

\[
\begin{align*}
\tilde{u}_0(x) &= u_0(x) & \text{for } x \in [0, \pi] \\
\tilde{u}_0(-x) &= -\tilde{u}_0(x) & \text{for } x \in \mathbb{R} \\
\tilde{u}_0(x+2\pi) &= \tilde{u}_0(x) & \text{for } x \in \mathbb{R}
\end{align*}
\]

Similar for \(v_0 \rightarrow \tilde{v}_0\).

Then, the solution is given by d'Alembert's formula:

\[
\begin{align*}
U(t,x) &= \frac{1}{2} \left( \tilde{u}_0(x+ct) + \tilde{v}_0(x-ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \tilde{v}_0(s) \, ds
\end{align*}
\]

Proof: Exercise

Solution via Fourier series:

If \(u_0 \in C^3(\mathbb{R})\) and \(v_0 \in C^2(\mathbb{R})\), then the solution is given by

\[
U(t,x) = \frac{2}{\pi} \sum_{k=0}^{\infty} \left( a_k \cos(kt) + b_k \sin(kt) \right) \sin(kx)
\]

with

\[
\begin{align*}
a_k &= \frac{2}{\pi} \int_{0}^{\pi} u_0(x) \sin(kx) \, dx \\
b_k &= \frac{2}{\pi} \int_{0}^{\pi} v_0(x) \sin(kx) \, dx
\end{align*}
\]

The higher regularity assumptions are required for uniform convergence of the Fourier series.
boundary conditions: \( x \in [0, \pi] \Rightarrow \sin(4x) = 0 \quad \checkmark \\
wave equation:
\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u(t, x) &= \frac{1}{c^2} \left( \sum_{k=1}^{\infty} \left( a_k c^2 \cos(kct) + b_k c^2 \sin(kct) \right) \sin(kx) \right) \\
\frac{\partial^2}{\partial x^2} u(t, x) &= -\frac{1}{c^2} \left( a_k \cos(kct) + b_k \sin(kct) \right) c^2 \sin(kx)
\end{align*}
\]
\( \checkmark \)
initial conditions \( \rightarrow \) fourier transform \( \checkmark \)

interpretation: decomposition into functions which oscillate in space and time. no convergence to a "steady state", no smoothing effect.

(c) non-differentiable solutions?

d'alambert's solution formulas can still be evaluated if \( u_0 \) and/or \( u_1 \) are not differentiable. in this case, the solution \( u(t, x) \) is not differentiable in space.

in which sense is such a function a "solution" of the wave equation?

\( \Rightarrow \) we have to define what we mean by "solution".
1.5 Some tools from operator theory

Motivation: Consider the ODE
\[ y' = Ay \]
\[ y(0) = y_0 \]
\( A \) is a linear operator, \( y(t) \in \mathbb{R}^d \)
\( v \in \mathbb{R}^d \) initial value

Solution:
\[ y(t) = e^{At} v \quad \text{with } e^{At} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \]

Now consider the PDE
\[ \Theta_t u(t,x) = \Theta_x u(t,x) \quad x \in \mathbb{R}, t \geq 0 \]
\[ u(0,x) = v(x) \]

Solution:
\[ \Theta_t u(t,x) = e^{t \Theta_x} v(x) \]

Problem: How can we define \( e^{t \Theta_x} \)? \[ \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \] does not converge if \( A \) is an unbounded operator.

Let \( X \) be a Banach space and let \( \mathbb{R}_+ := [0, \infty) \)

Definition (semigroup, generator)

A family \((T(t))_{t \in \mathbb{R}_+}\) of bounded operators \( T(t) : X \rightarrow X \) is called a \( C_0\)-semigroup (strongly continuous semigroup) if

(a) \( T(0) = I \) (identity)
\[ T(t+s) = T(t) T(s) \quad \forall t, s \in \mathbb{R}_+ \]

(b) For every \( v \in X \) the orbit map \( t \mapsto T(t) v \) is continuous \((\Leftarrow \lim_{t \searrow 0} T(t) v = v \quad \forall v \in X)\)

The generator of \((T(t))_{t \in \mathbb{R}_+}\) is the operator
\[ A v := \lim_{t \searrow 0} \frac{T(t) v - v}{t} \]
with domain \( D(A) := \{ v \in X : \lim_{t \searrow 0} \frac{T(t) v - v}{t} \text{ exists in } X \} \)
\((T_{\mathbb{R}^+})_{t \in \mathbb{R}^+}\) is a \(\mathbb{G}\)-group if "\(\mathbb{R}^+\) and "\(\mathbb{R}\)" can be replaced by \(\mathbb{R}\) and \(t \to 0\), respectively.

A (semi-)group is called
- bounded if there is a \(C \geq 1\) such that \(||T(t)||| \leq C \quad \forall t$
- contractive if \(||T(t)||| \leq 1 \quad \forall t$
- isometric if \(||T(t)||| = ||t||| \quad \forall t \quad \forall x \in X$
- unitary if \(T(1)\) is unitary for every \(t$

**Definition (ACP, classical solution)**

Let \(A : \mathcal{D}(A) \to X\) be a linear operator and \(v \in \mathcal{D}(A) \subseteq X\).

A function \(u : \mathbb{R}^+ \to X\) is a (classical) solution of the abstract Cauchy problem

\[
\begin{align*}
\forall t \in \mathbb{R}^+ & : \\
v(t) &= Au(t) \\
v(0) &= v
\end{align*}
\]

is \(u \in C^0(\mathbb{R}^+, X)\) \(\forall t \in \mathcal{D}(A) \forall t \in \mathbb{R}^+\) and (ACP) holds.

**Remark:** Typically \(X\) is a function space and \(A\) is a differential operator. For every \(t\), \(u(t)\) is a "point" in a function space, i.e. \(u(t) = u(t|x)\) is a function in time and space.

(ACP) is an ODE in a function space, i.e. a PDE.
Proposition 1.2

Let \((\mathbb{T}^t)_{t \in \mathbb{R}^+}\) be a \(C_0\)-semigroup with generator \(A: D(A) \rightarrow X\), \(D(A) \subseteq X\). Then if \(v \in D(A)\), then

\[ T(t)v \in D(A), \quad AT(t)v = T(t)Av \quad \forall t \in \mathbb{R}^+ \]

and

\[ u: \mathbb{R}^+ \rightarrow X, \quad u(t) = T(t)v \]

is the unique solution of \((ACP)\).

Definition (Mild Solution)

A continuous function \(u: \mathbb{R}^+ \rightarrow X\) is called a mild solution of \((ACP)\) if

\[ \frac{1}{s} \int_{0}^{s} u(s)ds \in D(A) \quad \text{and} \quad u(t) = v + \int_{0}^{t} A(s)u(s)ds. \]

Proposition 1.3

\(u(t)\) classical solution of \((ACP) \Rightarrow u(t)\) mild solution of \((ACP)\).

For every \(v \in X\), \(u(t) := T(t)v\) is the unique mild solution of \((ACP)\).

Definition (Closed Operator)

A linear operator \(A: D(A) \rightarrow X\) with \(D(A) \subseteq X\) is closed if it has one of the following equivalent properties:

(a) If \((v_n)_{n=1}^{\infty}\) is a sequence in \(D(A)\) with \(\lim_{n \rightarrow \infty} v_n = v \in X\)

\[ \lim_{n \rightarrow \infty} Av_n = v \in X \]

then \(v \in D(A)\) and \(Av = v\).

(b) \(D(A)\) with norm \(\|\cdot\|_X\) is a Banach space, where \(\|u\| := \|u\| + \|Au\|\) is the graph norm of \(A\).