Numerical methods in mathematical finance
Winter Semester 2012/13

Problem 1  (Trading strategies involving options)
Combinations of vanilla options can be used for more effective speculation/hedging in realistic market environments. Derive a formula for the value of each of the following trading strategies and sketch the corresponding payoff diagrams.

(a) Straddle: hold a call option and a put option on the same underlying asset with the same maturity date $T$ and same strike price $K$.

(b) Strangle: hold a call option and a put option on the same underlying asset with the same maturity date $T$ and different strike prices. The call strike $K_2$ is higher than the put strike $K_1$.

(c) Butterfly spread: hold a European call option with strike price $K_1$ and another with strike price $K_3$, where $K_3 > K_1$ and also write two calls with strike price $K_2 = (K_1 + K_3)/2$.

Solution:
Each option contributes towards the overall value of the portfolio at maturity.

(a) Holding a call and a put with the same maturity date $T$ and same strike price $K$ means that the value of the portfolio at maturity is

$$\pi_T = (K - S)^+ + (S - K)^+ = \begin{cases} S - K & : S > K \\ K - S & : S \leq K \end{cases} \equiv |S - K|.$$ 

A straddle is appropriate when the holder is expecting large movements in either direction, as it produces a profit if the strike price $K$ differs significantly from the price of the underlying asset. If the price of the underlying is close to $K$, the strategy leads to a loss.

(b) A strangle is similar to a straddle, as the investor is betting that there will be a large price move. At maturity, the value of the portfolio is given by

$$\pi_T = (K_1 - S)^+ + (S - K_2)^+.$$ 

The payoff diagram depends on how close the strike prices are. The farther they are apart, the bigger the price movement of the underlying in order for the strategy to turn a profit. The strangle has however less risk for the investor if the price ends up near the middle value. Because the chances of turning a profit are smaller, a strangle is cheaper than a straddle.

(c) Holding (buying) two European call options with strike prices $K_1$ and $K_3$ and writing (selling) two European call options with strike price $K_2 = (K_1 + K_3)/2$ means that the value of the portfolio at maturity is given by

$$\pi_T = (S - K_1)^+ + (S - K_3)^+ - (S - K_2)^+ - (S - K_2)^+$$

The payoff of the butterfly spread is investigated in Table 2. Note that the payoff curve attains its maximum value of $(K_3 - K_1)/2$ at $S = K_2$.

<table>
<thead>
<tr>
<th>Underlying price range</th>
<th>Payoff call</th>
<th>Payoff put</th>
<th>Total payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S \leq K_1$</td>
<td>0</td>
<td>$K_1 - S$</td>
<td>$K_1 - S$</td>
</tr>
<tr>
<td>$K_1 &lt; S &lt; K_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S \geq K_2$</td>
<td>$S - K_2$</td>
<td>0</td>
<td>$S - K_2$</td>
</tr>
</tbody>
</table>

A butterfly spread is useful for cases where the underlying is expected to stagnate around the price $K_2$, or have only small oscillations. If European call options with maturity $T$ were to be available for every possible strike price, it would in theory be possible to approximate any payoff function using a combination of butterfly spreads where each pair of values $K_1$ and $K_3$ are chosen to be close together.
**Problem 2** (Put-Call parity)
Assume an idealized market satisfying assumptions (A1)-(A5) from the Lecture. Consider a portfolio \( \pi \) made up of a stock currently worth 75 €, and the corresponding European call and put options with maturity of 1 year and strike price 70 €. Knowing that the call option costs 9 € more than the put option, determine the current risk-free interest rate \( r > 0 \).

**Solution:**
Use the Put-Call parity
\[
S(t) + V_P(t, S(t)) - V_C(t, S(t)) = e^{-r(T-t)} K
\]
Inserting the values into the expression above we obtain:
\[
75 - 9 = 70e^{-r} \iff 66 = 70e^{-r} \iff 33 = e^{-r} \iff r = \log \left( \frac{35}{33} \right) \approx 0.0588
\]
The interest rate is therefore \( r \approx 5.88\% \).

**Problem 3** (Upper and lower bounds on option prices)
Consider a liquid financial market with a risk-free interest rate \( r > 0 \), where divisibility holds and arbitrage is impossible. Further, consider an European put option with strike price \( K > 0 \) and maturity \( T \) on an underlying asset with price \( S = S(t) \geq 0, t \in [0, T] \), and let \( V_P(t, S) \) denote its value. Show that the following inequalities hold
\[
(Ke^{-r(T-t)} - S)^+ \leq V_P(t, S) \leq Ke^{-r(T-t)} \quad \forall t \in [0, T]
\]
**Hint:** Use the bounds on the European call option price from the Lecture and the put-call parity.

**Solution:**
Using put call parity and the bounds on the call option price from Lemma 1.4.2 from the Lecture, we first have that the value of a call option with the same underlying, strike price and maturity \( T \) satisfies
\[
V_C(t, S) = S + V_P(t, S) - Ke^{-r(T-t)} \leq S,
\]
therefore,
\[
V_P(t, S) - Ke^{-r(T-t)} \leq 0 \iff V_P(t, S) \leq Ke^{-r(T-t)}.
\]
From the proof of Lemma 1.4.2, we have that
\[
V_C(t, S) = S + V_P(t, S) - Ke^{-r(T-t)} \geq (S - Ke^{-r(T-t)})^+.
\]
For the case \( S \geq Ke^{-r(T-t)} \) we thus obtain that
\[
S + V_P(t, S) - Ke^{-r(T-t)} \geq S - Ke^{-r(T-t)} \iff V_P(t, S) \geq 0.
\]
For the case \( S < Ke^{-r(T-t)} \) on the other hand, we obtain
\[
S + V_P(t, S) - Ke^{-r(T-t)} \geq 0 \iff V_P(t, S) \geq Ke^{-r(T-t)} - S.
\]
Summarizing, we have obtained that \( V_P(t, S) \geq (Ke^{-r(T-t)} - S)^+ \).

**Problem 4** (Arbitrage)
While reading the financial section of his morning paper, Mr. T learns that the value of call and put options with strike price 50 € and maturity of 1 year on a stock currently worth 50 €, is 5 € and 4 €, respectively. Knowing that 45 € invested in a bond are worth 50 € after 1 year, Mr. T smiles while calling his bank advisor. Show that Mr. T does indeed have reasons to smile. Is the described scenario valid in the idealized market introduced in the Lecture? (supply a rigorous argument).

**Solution:**
Use the put-call parity to observe that
\[
50 + 4 - 5 = 45 \iff 49 = 45 !
\]
We construct the following strategy which will result in arbitrage:
- Invest 45 € in a bond with interest rate \( r \)
- Buy one call option for \( V_C = 5 € \)
- Short sell one stock for \( S(0) = 50 € \)
- Write and sell a put option for \( V_P = 4 € \)

These actions are summarized in the arbitrage Table 3.

**Note:** short selling means that an investor borrows a stock, sells it and uses the money for other investments. However, he must also be able to close the position, i.e., keep enough funds to repurchase the stock and return it to its owner. If the investor has taken a short position on some assets in his portfolio, their contribution to the portfolio value will be negative. Otherwise, if a long position has been taken, the value of the assets will be registered as being positive.

At time \( T \), the investor will
- collect 50 € from his bond investment
Table 3: Arbitrage table for Problem 4

<table>
<thead>
<tr>
<th>Action</th>
<th>Money flow</th>
<th>Portfolio value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t = 0$</td>
<td>$t = T$</td>
</tr>
<tr>
<td>Buy Bond</td>
<td>$-45$</td>
<td>$45$</td>
</tr>
<tr>
<td>Buy Call</td>
<td>$-5$</td>
<td>$5$</td>
</tr>
<tr>
<td>Sell Stock</td>
<td>$50$</td>
<td>$-50$</td>
</tr>
<tr>
<td>Sell Put</td>
<td>$4$</td>
<td>$-4$</td>
</tr>
<tr>
<td>Sum</td>
<td>$4$</td>
<td>$-4$</td>
</tr>
</tbody>
</table>

- buy one stock at current market value for $S(T)$ and either exercise the call option if $S(T) > 50$ or settle the put option if $S(T) \leq 50$. The balance of these transactions is 0.

At $t = 0$ the balance is $-4$, which means this arbitrage strategy generates an immediate risk-free profit of 4 €!

**Problem 5** (Arbitrage)

Let $C_1$ and $C_2$ denote the values of two European call options on the same underlying and with the same maturity $T = 1$ year, but with different strike prices, $K_1 = 47.7 \text{ €}$ and $K_2 = 40 \text{ €}$, respectively. Using the risk-free interest rate $r = 10\%$ and the values at $t = 0$ of the two options, $C_1(0) = 5.2 \text{ €}$ and $C_2(0) = 12.4 \text{ €}$, devise an arbitrage strategy.

**Solution:**

We construct an arbitrage strategy:

<table>
<thead>
<tr>
<th>Action</th>
<th>$t = 0$ $S(T) \leq 40$</th>
<th>$t = T$ $40 &lt; S(T) \leq 47.7$</th>
<th>$S(T) &gt; 47.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Bonds</td>
<td>$7$</td>
<td>$7.7$</td>
<td>$7.7$</td>
</tr>
<tr>
<td>Buy $C_1$</td>
<td>$5.2$</td>
<td>$0$</td>
<td>$S(T) - 47.7$</td>
</tr>
<tr>
<td>Sell $C_2$</td>
<td>$-12.4$</td>
<td>$0$</td>
<td>$40 - S(T)$</td>
</tr>
<tr>
<td>Portfolio</td>
<td>$-0.2$</td>
<td>$7.7$</td>
<td>$47.7 - S(T) \geq 0$</td>
</tr>
</tbody>
</table>

We obtain at $t = 0$ an immediate profit of 0.2 €.

This sheet will be discussed in the problem sessions on **26th October, 2012**.