Problem 1 (Moments of GBM)
A stochastic process $S_t$ is called a geometric Brownian motion (GBM) if it satisfies the SDE
\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \]
with constants $\mu, \sigma \in \mathbb{R}$. For a fixed initial value $S_0$, the above SDE has the solution
\[ S_t = S_0 \exp(at + \sigma W_t), \quad a = \mu - \frac{\sigma^2}{2}. \]

Show that $S_t$ has the following properties:
(a) $E(S_t) = S_0 e^{at}$
(b) $E(S_t^2) = S_0^2 e^{2(\mu + \sigma^2)t}$
(c) $\text{Var}(S_t) = S_0^2 e^{2at} \left(e^{\sigma^2t} - 1\right)$

Solution:

(a) Using the analytical solution, we compute
\[
E(S_t) = E(S_0 e^{(at + \sigma W_t)}) = S_0 e^{at} E(e^\sigma W_t)
\]
We know that $W_t = W_t - W_0 \sim \mathcal{N}(0, t)$, and thus obtain
\[
E(e^\sigma W_t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^\sigma x e^{-\frac{x^2}{2t}} dx.
\]
Note that
\[
\frac{(x - \sigma t)^2}{2t} = -\frac{1}{2t}(x^2 - 2x\sigma t + \sigma^2 t^2) = -\frac{x^2}{2t} + \sigma x - \frac{\sigma^2 t}{2},
\]
which means we can make the substitution
\[
e^{\sigma x} e^{-\frac{x^2}{2t}} = e^{-\frac{(x - \sigma t)^2}{2t}} e^{-\frac{x^2}{2t}},
\]
leading to
\[
E(e^\sigma W_t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x - \sigma t)^2}{2t}} e^{-\frac{x^2}{2t}} dx = e^{\frac{\sigma^2 t}{2}} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x - \sigma t)^2}{2t}} dx = e^{\frac{\sigma^2 t}{2}}.
\]
The last result follows from the fact that $\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x - \sigma t)^2}{2t}}$ is the density function of the Gaussian distribution $\mathcal{N}(\sigma t, t)$. Lastly,
\[
E(S_t) = S_0 e^{at} e^{\frac{\sigma^2 t}{2}} = S_0 e^{at} (e^{\frac{\sigma^2 t}{2}}) = S_0 e^{at}.
\]

(b) We have
\[
E(S_t^2) = E\left((S_0 e^{(at + \sigma W_t)})^2\right) = (S_0 e^{at})^2 E(e^{2\sigma W_t})
\]
Similar to (a), but using $2\sigma$ instead of $\sigma$, we can compute
\[
E(e^{2\sigma W_t}) = e^{2\sigma^2 t}.
\]
Thus,
\[
E(S_t^2) = (S_0 e^{at})^2 e^{2\sigma^2 t} = S_0^2 e^{2at} e^{2\sigma^2 t} = S_0^2 e^{2at} (e^{\sigma^2 t} - 1) = S_0^2 e^{2at} (e^{\sigma^2 t} - 1).
\]

(c) Using the expression for variance and the properties (a) and (b), we get
\[
\text{Var}(S_t) = E(S_t^2) - (E(S_t))^2 = S_0^2 e^{2at} (e^{\sigma^2 t} - 1) - S_0^2 e^{2at} = S_0^2 e^{2at} (e^{\sigma^2 t} - 1).
\]
**Problem 2** (Log-normal distribution)

The geometric Brownian motion

\[ S_t = S_0 \exp(at + \sigma W_t), \quad a = \mu - \frac{\sigma^2}{2} \]

has log-normal distribution, i.e.,

\[ \ln S_t = \ln S_0 + at + \sigma W_t \sim \mathcal{N}(\ln S_0 + at, \sigma^2 t). \]

Show that \( S_t \) has density function

\[ \phi(x) = \phi(x, \tilde{\mu}, \tilde{\sigma}) = \begin{cases} 
\frac{1}{\sqrt{2\pi} \tilde{\sigma} x} \exp \left( -\frac{(\ln x - \tilde{\mu})^2}{2\tilde{\sigma}^2} \right), & \text{if } x > 0 \\
0, & \text{else} 
\end{cases} \]

with \( \tilde{\mu} = \ln S_0 + at \) and \( \tilde{\sigma} = \sigma \sqrt{t} \).

**Hint:** Use that for a continuous random variable \( X \),

\[ P(u \leq X \leq v) = \int_u^v \phi(x) \, dx, \]

where \( \phi \) is the corresponding probability density function.

**Solution:**

Using the fact that \( W_t \sim \mathcal{N}(0, t) \), we get

\[ P(u \leq S_t \leq v) = P(u \leq S_0 \exp(at + \sigma W_t) \leq v) = P \left( \frac{\ln(u/S_0) - at}{\sigma} \leq W_t \leq \frac{\ln(v/S_0) - at}{\sigma} \right) = \frac{1}{\sqrt{2\pi} t} \int_{\frac{\ln(u/S_0) - at}{\sigma}}^{\frac{\ln(v/S_0) - at}{\sigma}} e^{-\frac{s^2}{2}} \, ds 
\]

Next, using the substitution

\[ x = S_0 e^{at + \sigma s}, \]

which leads to \( ds = \frac{1}{\sigma x} \, dx \), we can write the integral as

\[ \frac{1}{\sqrt{2\pi} t} \int_u^v \frac{e^{-\frac{(\ln(x/S_0) - at)^2}{2\sigma^2}}}{\sigma x} \, dx, \]

and deduce that the corresponding density function \( \phi(x) \) for \( x > 0 \) is

\[ \phi(x) = \frac{1}{\sqrt{2\pi} \tilde{\sigma} x} e^{-\frac{(\ln(x/S_0) - at)^2}{2\tilde{\sigma}^2}} = \frac{1}{\sqrt{2\pi} \tilde{\sigma} x} e^{-\frac{(\ln x - \tilde{\mu})^2}{2\tilde{\sigma}^2}}. \]

**Problem 3** (Solution of the Heat equation)

Let \( u_0 : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function which satisfies the growth condition

\[ |u_0(x)| \leq Me^{\gamma x^2}, \]

with constants \( M > 0 \) and \( \gamma \in \mathbb{R} \). Then, the function

\[ u(\tau, x) := \frac{1}{\sqrt{4\pi \tau}} \int_{-\infty}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4\tau} \right) u_0(\xi) \, d\xi, \]

is a solution of the heat equation

\[ \partial_\tau u(\tau, x) = \partial_x^2 u(\tau, x), \quad \text{for } x \in \mathbb{R}, \tau > 0, \]

and we have

\[ \lim_{\tau \to 0} u(\tau, x) = u_0(x). \]
**Hint:** Compute the partial derivatives and substitute into the heat equation. To check the last assertion use the transformation $\eta = (\xi - x)/\sqrt{4\tau}$.

**Solution:**

We compute

$$
\partial_x u(\tau, x) = -\frac{1}{4\sqrt{\tau}} \int_{-\infty}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4\tau} \right) u_0(\xi) \, d\xi
+ \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4\tau} \right) u_0(\xi) \, d\xi
= \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4\tau} \right) u_0(\xi) \, d\xi
- \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4\tau} \right) u_0(\xi) \, d\xi
$$

and by direct comparison, we get

$$
\partial_x u(\tau, x) = \partial_x^2 u(\tau, x).
$$

Performing the substitution

$$
\eta := \frac{\xi - x}{2\sqrt{\tau}}, \quad d\eta = \frac{1}{2\sqrt{\tau}} \, d\xi,
$$

we obtain

$$
\lim_{\tau \to 0} u(\tau, x) = \lim_{\tau \to 0} \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{\infty} \exp(-\eta^2) u_0(x + 2\eta \sqrt{\tau}) 2\sqrt{\tau} \, d\eta
= \lim_{\tau \to 0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\eta^2) u_0(x + 2\eta \sqrt{\tau}) \, d\eta
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\eta^2) u_0(x) \, d\eta
= \frac{1}{\sqrt{\pi}} u_0(x) \int_{-\infty}^{\infty} \exp(-\eta^2) \, d\eta
= u_0(x).
$$

**Problem 4** (Hedging portfolios: The Greeks)

Let $V(t, S)$ be the value of an European call as given by the Black-Scholes equation. The sensitivity of the option price to the underlying asset price $S$ can be measured by computing Delta,

$$
\Delta = \frac{\partial V}{\partial S},
$$

which is one of the so-called Greeks. Show that

$$
\Delta = \Phi(d_1) > 0,
$$

where we have by the Black-Scholes formula for calls,

$$
V(t, S) = S \Phi(d_1) - K \exp(-r(T-t)) \Phi(d_2)
$$

with

$$
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} \, ds,
$$

$$
\Phi'(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},
$$

$$
d_{1/2} = \frac{\ln \frac{S_k}{K} + \left( r + \frac{\sigma^2}{2} \right) (T-t)}{\sigma \sqrt{T-t}}.
$$

**Solution:**

$$
\Delta = \frac{\partial V}{\partial S} = \Phi(d_1) + S \Phi'(d_1) \frac{\partial d_1}{\partial S} - K \exp(-r(T-t)) \Phi'(d_2) \frac{\partial d_2}{\partial S}
$$

The partial derivatives $\frac{\partial d_1}{\partial S}$ and $\frac{\partial d_2}{\partial S}$ can be easily computed from the definition of $d_{1/2}$, and we have

$$
\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S \sigma \sqrt{T-t}}.
$$

Also note the relation

$$
d_2 = \frac{\ln \frac{S}{K} + \left( r - \frac{\sigma^2}{2} \right) (T-t)}{\sigma \sqrt{T-t}}
= \frac{\ln \frac{S}{K} + \left( r - \frac{\sigma^2}{2} - \sigma^2 \right) (T-t)}{\sigma \sqrt{T-t}}
= \frac{d_1 - \sigma \sqrt{T-t}}{\sigma \sqrt{T-t}}.
$$
Thus,
\[ \Delta = \Phi(d_1) + \frac{1}{\sigma \sqrt{T-t}} \left( \phi(d_1) - \frac{K}{S} \exp(-r(T-t))\phi(d_2) \right). \]

Next, we show that
\[ \phi(d_1) - \frac{K}{S} \exp(-r(T-t))\phi(d_2) = 0, \tag{⋆} \]
by considering
\[ \ln \left( \frac{S\phi(d_1)}{K \exp(-r(T-t))\phi(d_2)} \right) = \ln(S/K) + r(T-t) + \ln \left( \frac{\phi(d_1)}{\phi(d_2)} \right), \tag{★★} \]
and the definition of the density function \( \phi \), which leads to the simplification of the last term to
\[ \ln \left( \frac{\phi(d_1)}{\phi(d_2)} \right) = -\frac{1}{2}(d_1^2 - d_2^2). \]

Next, by using the relation between \( d_1 \) and \( d_2 \), we get
\[ \begin{align*}
d_1^2 - d_2^2 &= d_1^2 - (d_1 - \sigma \sqrt{T-t})^2 \\
&= 2d_1 \sigma \sqrt{T-t} - \sigma^2(T-t) \\
&= 2 \ln \frac{S}{K} + 2r(T-t).
\end{align*} \]

Therefore,
\[ \ln \left( \frac{\phi(d_1)}{\phi(d_2)} \right) = -\ln(S/K) - r(T-t), \]
and by plugging the expression into (★★), we obtain
\[ \ln \left( \frac{S\phi(d_1)}{K \exp(-r(T-t))\phi(d_2)} \right) = 0, \]
which is equivalent to (⋆). Summarizing, we have obtained that the \( \Delta \) for an European call option is simply,
\[ \Delta = \Phi(d_1) > 0. \]

**Discussion:** The Greeks are tools for risk-management. Because delta measures the exposure of the option price to the price of the underlying asset, a portfolio with value II is hedged when \( \Delta \Pi = \partial \Pi / \partial S = 0 \), meaning its value will not change for small changes in the price of the underlying. Delta-hedging refers to the strategy where by buying or selling some amount of the underlying, the portfolio is made delta neutral, i.e., its \( \Delta \Pi = 0 \). The delta-hedge must be readjusted periodically in order to maintain \( \Delta \Pi = 0 \).

Considering a portfolio containing bonds, stock and a hedged option (as in Section 3.2 in the Lecture), its value is given by
\[ \Pi = -V(t, S) + a_t S + b_t B_t. \]

Assuming bonds with current value 1 are used, we have
\[ \Delta \Pi = \partial S \Pi = - \partial S V(t, S) + a_t, \]
and from the condition \( \Delta \Pi = 0 \) it follows that \( a_t = \partial S V(t, S) = \Delta = \Phi(d_1). \)

This sheet will be discussed in the problem sessions on **30th November, 2012**.