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Wavelets – Theory and Applications (Winter 2008/09)

3.3.4.4 Daubechies wavelets

B-splines have the very simple symbol $(\frac{1+e^{-\imath\omega}}{2})^N$ which is a trigonometric polynomial, unfortunately, not satisfying property (Prop. O).

Daubechies considered a modification:

$$H_N(\omega) = \Big(\frac{1+e^{-\imath\omega}}{2}\Big)^N L(e^{-\imath\omega}) = 2^{-1/2}\sum_{k=0}^{K_N-1} h_k e^{-\imath k\omega}, \quad h_k \in \mathbb{R},\ K_N \geq N+1,$$

where L is a polynomial of degree $K_N - N - 1$ with $L(1) = 1$ (yielding $H_N(0) = 1$) and $L(e^{-\imath\pi}) = L(-1) \neq 0$.

Now the task is to determine L such that (Prop. O) is fulfilled. The identity

$$\Big|\frac{1+e^{-\imath\omega}}{2}\Big|^2 = \cos^2\Big(\frac{\omega}{2}\Big) = \frac{1+\cos\omega}{2}$$

gives

$$|H_N(\omega)|^2 = \cos^{2N}\Big(\frac{\omega}{2}\Big)\,|L(e^{-\imath\omega})|^2.$$

As H_N has real coefficients we find

$$|H_N(\omega)|^2 = \overline{H_N(\omega)}H_N(\omega) = H_N(-\omega)H_N(\omega)$$

revealing $|H_N(\omega)|^2$ to be a polynomial in $\cos\omega$. Thus, $|L(e^{-\imath\omega})|^2$ must be a polynomial in $\cos\omega = 1 - \sin^2(\frac{\omega}{2})$ as well, that is,

$$|L(e^{-\imath\omega})|^2 = P\Big(\sin^2\Big(\frac{\omega}{2}\Big)\Big)$$

where P is a polynomial. Thus,

$$|H_N(\omega)|^2 + |H_N(\omega+\pi)|^2 = \cos^{2N}\Big(\frac{\omega}{2}\Big)\,P\Big(\sin^2\Big(\frac{\omega}{2}\Big)\Big) + \sin^{2N}\Big(\frac{\omega}{2}\Big)\,P\Big(\cos^2\Big(\frac{\omega}{2}\Big)\Big).$$

Set $x := \sin^2(\frac{\omega}{2})$. Then, we need to find a polynomial P with $P|_{[0,1]} \geq 0$ such that Bezout's equation

$$(1-x)^N P(x) + x^N P(1-x) = 1$$

is satisfied.

Lemma 3.27 (Bezout)
We have that

$$P_N(x) = \sum_{k=0}^{N-1}\binom{N-1+k}{k}x^k \geq 0, \quad x\in[0,1],$$

is the unique minimal degree solution of Bezout's equation.

As a consequence we find that

$$|H_N(\omega)|^2 = \cos^{2N}\Big(\frac{\omega}{2}\Big)\, P_N\Big(\sin^2\Big(\frac{\omega}{2}\Big)\Big) = \Big(\frac{1+\cos\omega}{2}\Big)^N P_N\Big(\frac{1-\cos\omega}{2}\Big)$$

The ansatz $H_N(\omega) = 2^{-1/2}\sum_{k=0}^{2N-1} h_k e^{-\imath k\omega}$, that is $K_N = 2N$, leads to

$$|H_N(\omega)|^2 = 2^{-1}\sum_{r=-(2N-1)}^{2N-1} d_r e^{-\imath r\omega} \quad \text{with } d_r = \sum_{k=\max\{-r,0\}}^{\min\{2N-1,2N-1-r\}} h_{r+k}h_k.$$

By symmetry ($d_{-r} = d_r$) we obtain

$$|H_N(\omega)|^2 = \frac{1}{2}d_0 + \sum_{r=1}^{2N-1} d_r\cos(r\omega) \stackrel{!}{=} \Big(\frac{1+\cos\omega}{2}\Big)^N P_N\Big(\frac{1-\cos\omega}{2}\Big).$$

Equating coefficients yields a nonlinear system of equations for $\{h_k\}_{0\le k\le 2N-1}$ which can be solved numerically.

Let us look at two examples:
<u>$N = 1$:</u> From

$$\Big(\frac{1+\cos\omega}{2}\Big)\underbrace{P_1\Big(\frac{1-\cos\omega}{2}\Big)}_{=1} = \frac{1}{2} + \frac{1}{2}\cos\omega \stackrel{!}{=} \frac{1}{2}d_0 + d_1\cos\omega$$

we deduce the system

$$h_0^2 + h_1^2 = d_0 = 1 \quad \text{and} \quad h_0h_1 = d_1 = \frac{1}{2}.$$

having the solutions

$$h_0 = h_1 = \pm\frac{1}{\sqrt{2}}.$$

The normalization $H_1(0) = 1$, that is, $h_0+h_1 = \sqrt{2}$ yields the unique solution $h_0 = h_1 = 2^{-1/2}$ leading to the Haar system.
<u>$N = 2$:</u> Here,

$$\frac{(1+\cos\omega)^2}{4}\underbrace{P_2\Big(\frac{1-\cos\omega}{2}\Big)}_{=2-\cos\omega} = \frac{1}{2} + \frac{9}{16}\cos\omega - \frac{1}{16}\cos(3\omega)$$
$$\stackrel{!}{=} \frac{1}{2}d_0 + d_1\cos\omega + d_2\cos(2\omega) + d_3\cos(3\omega).$$

Equating coefficients yields

$$d_0 = 1, \quad d_1 = \frac{9}{16}, \quad d_2 = 0, \quad d_3 = -\frac{1}{16},$$

and

$$\begin{aligned} h_0^2 + h_1^2 + h_2^2 + h_3^2 &= 1, \\ h_1h_0 + h_2h_1 + h_3h_2 &= \frac{9}{16}, \\ h_2h_0 + h_3h_1 &= 0, \\ h_3h_0 &= -\frac{1}{16}. \end{aligned}$$

Above system has the two solutions

$$h_0 = \frac{1 \pm \sqrt{3}}{4\sqrt{2}}, \quad h_1 = \frac{3 \pm \sqrt{3}}{4\sqrt{2}}, \quad h_2 = \frac{3 \mp \sqrt{3}}{4\sqrt{2}}, \quad h_3 = \frac{1 \mp \sqrt{3}}{4\sqrt{2}},$$

satisfying $H_2(0) = 1$.

The number of normalized ($H_N(0) = 1$) solutions of the nonlinear system increases with N. Can we characterize a special solution?

Recall that

$$|L(\mathrm{e}^{-\imath\omega})|^2 = L(\mathrm{e}^{-\imath\omega})L(\mathrm{e}^{\imath\omega}) = P_N\Big(\frac{1-\cos\omega}{2}\Big) = P_N\Big(\frac{2-\mathrm{e}^{-\imath\omega}-\mathrm{e}^{\imath\omega}}{4}\Big).$$

This is a factorization problem which we consider on the whole complex plane:

$$L(z)L(z^{-1}) = r_0^2 \prod_{k=1}^{N-1}(1-a_kz)(1-a_kz^{-1}) \overset{!}{=} P_N\Big(\frac{2-(z+z^{-1})}{4}\Big) =: Q(z)z^{-(N-1)}$$

where $r_0^{-1} = \prod_{k=1}^{N-1}(1-a_k)$ (guaranteeing $L(1) = 1$) and where Q is polynomial of degree $2(N-1)$.

Since Q has only real coefficients all of its roots are conjugate: if $Q(c) = 0$ then $Q(\overline{c}) = 0$ as well. Also, if $Q(c) = 0$ then $Q(c^{-1}) = 0$. Thus, Q can be factorized as

$$Q(z) = \underbrace{\prod_{l=1}^{M}(z-z_l)(z-\overline{z}_l)(z-z_l^{-1})(z-\overline{z}_l^{-1})}_{\text{complex roots}}\underbrace{\prod_{k=1}^{R}(z-r_k)(z-r_k^{-1})}_{\text{real roots}}$$

where $4M + 2R = 2(N-1)$.

For the construction of L we choose the a_k's as roots of Q such that $|a_k| \le 1$ (this choice is always possible since the roots of Q come in reciprocal pairs). If a_k is a complex root of Q we also take $\overline{a}_k$ as root of L to guarantee real coefficients. The resulting filters $\{h_k\}_{0\le k\le 2N-1}$ are the *Daubechies filters*. They are unique up to ordering.

Let us reconsider the situation for $N = 2$. Here,

$$Q(z) = zP_2\Big(\frac{2-(z+z^{-1})}{4}\Big) = -\frac{1}{2}z^2 + 2z - \frac{1}{2}.$$

The roots are $2+\sqrt{3}$ and $2-\sqrt{3} = (2+\sqrt{3})^{-1}$. Hence,

$$L(z) = r_0(1-a_1z) \quad \text{with } a_1 = 2-\sqrt{3}\ (|a_1| \le 1) \text{ and } r_0 = \frac{1}{1-a_1} = \frac{1}{\sqrt{3}-1},$$

that is,

$$L(z) = \frac{\sqrt{3}+1}{2} - \frac{\sqrt{3}-1}{2}\, z.$$

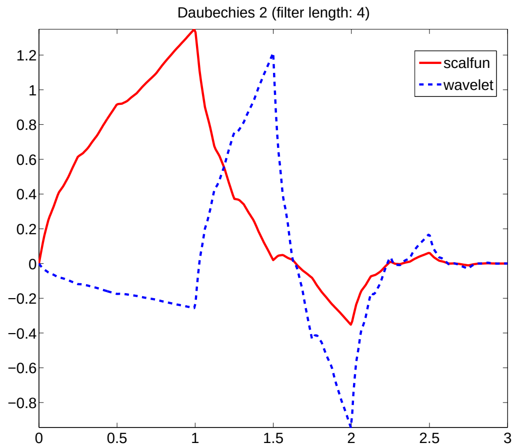
Finally,

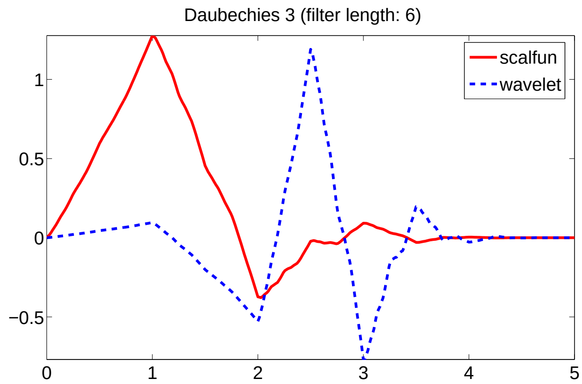
$$H_2(z) = \left(\frac{1+z}{2}\right)^2 L(z) = \frac{1}{8}\big(\sqrt{3}+1+(3+\sqrt{3})z+(3-\sqrt{3})z^2+(1-\sqrt{3})z^3\big)$$

resulting in

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, \quad h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, \quad h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

The corresponding wavelet system is shown on the left below.





Symmlets

All Daubechies wavelets are rather asymmetric, a property which can even be verified. The reason is that we always choose roots of Q within the unit circle to construct L. Daubechies has shown that there is no factorization of Q leading to symmetric wavelets and scaling functions (except for the Haar wavelet). It is, however, possible to factorize Q such that the scaling functions are “least” asymmetric.

A function $a(\xi) = \sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{-\imath n\xi}$, $a_n \in \mathbb{R}$, has *linear phase* if

$$a(\xi) = \mathrm{e}^{-\imath l\xi}|a(\xi)| \text{ for one } l \in \mathbb{Z}.$$

Linear phase implies symmetry with respect to l: $a_{l-n} = a_{l+n}$. Indeed,

$$a_{l-n} = \frac{1}{2\pi}\int_0^{2\pi} a(\xi)\mathrm{e}^{\imath(l-n)\xi}\mathrm{d}\xi = \frac{1}{2\pi}\int_0^{2\pi} |a(\xi)|\mathrm{e}^{-\imath n\xi}\mathrm{d}\xi$$

and

$$a_{l+n} = \frac{1}{2\pi}\int_0^{2\pi} a(\xi)\mathrm{e}^{\imath(l+n)\xi}\mathrm{d}\xi = \frac{1}{2\pi}\int_0^{2\pi} |a(\xi)|\mathrm{e}^{\imath n\xi}\mathrm{d}\xi$$

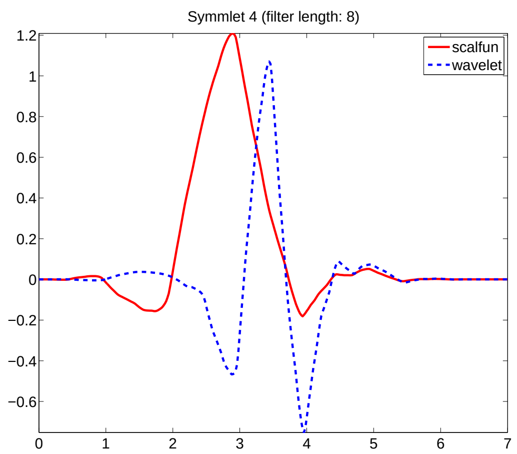
yield

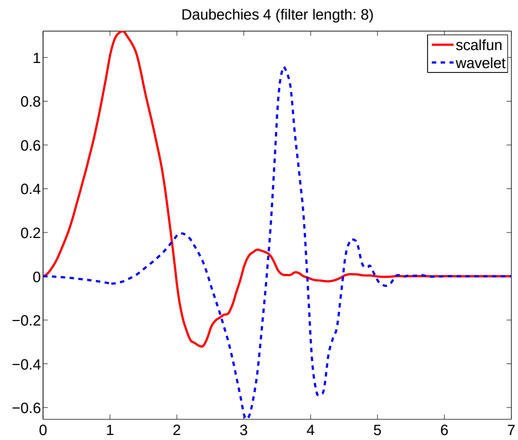
$$a_{l+n} = \overline{a_{l-n}} = a_{l-n}.$$

Idea: For the construction of L select the roots of Q such that the symbol H_N is as close to linear phase as possible.
For any choice of the roots we can compute the pase Φ_N of H_N. The symmlet filter is the one minimizing

$$\Psi_N(\xi) = \Phi_N(\xi) - \xi \frac{\Phi_N(2\pi)}{2\pi}.$$

The corresponding wavelets are called *symmlets*.





Coiflets

In all our examples so far, $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ constitutes an ONB in V_j:

$$P_j f = \sum_{k\in\mathbb{Z}} \underbrace{\langle f, \varphi_{j,k}\rangle_{L^2}}_{= c_{j,k}(f)} \varphi_{j,k}$$

is the orthogonal projection onto V_j.

Assume that we only know discrete samples $\{f(2^{-j}k)\}_{k\in\mathbb{Z}}$ of f to compute $P_j f$. How can we proceed?

1. Possibility: Construct adapted quadrature rules to approximate $\langle f, \varphi_{j,k}\rangle_{L^2}$.
2. Possibility: Suppose that $\int_{\mathbb{R}} \varphi(x)\mathrm{d}x = 1$ and $\int_{\mathbb{R}} x^k \varphi(x)\mathrm{d}x = 0$, $k = 1, \ldots, L-1$. Then, by the Taylor expansion

$$f(x) = \sum_{m=0}^{L-1} \frac{f^{(m)}(2^{-j}k)}{m!}(x - 2^{-j}k)^m + \frac{1}{(L-1)!}\int_{2^{-j}k}^{x} (x-t)^{L-1} f^{(L)}(t)\mathrm{d}t$$

we obtain

$$\langle f, \varphi_{j,k}\rangle_{L^2} = 2^{-j/2} f(2^{-j}k) + \frac{1}{(L-1)!}\int_{I_{j,k}} \int_{2^{-j}k}^{x} (x-t)^{L-1} f^{(L)}(t)\mathrm{d}t \, \varphi_{j,k}(x)\mathrm{d}x$$

where $I_{j,k} = \operatorname{supp} \varphi_{j,k}$. Hence,

$$\left|\langle f, \varphi_{j,k}\rangle_{L^2} - 2^{-j/2} f(2^{-j}k)\right| \le C_L \|f^{(L)}\|_{L^2(I_{j,k})} 2^{-jL},$$

that is, $2^{-j/2} f(2^{-j}k)$ approximates $\langle f, \varphi_{j,k}\rangle_{L^2}$ with a high order of accuracy.

To an even number $M \in 2\mathbb{Z}$ Daubechies constructed conjugate mirror filters of length $3M$ such that the corresponding orthogonal scaling function φ_M and wavelet ψ_M satisfy: $\operatorname{supp}\varphi_M = \operatorname{supp}\psi_M = [0, 3M-1]$,

$$\int_{\mathbb{R}} \varphi_M(x)\mathrm{d}x = 1, \quad \int_{\mathbb{R}} x^l \varphi_M(x)\mathrm{d}x = 0, \; l = 1, \dots M-1,$$

and

$$\int_{\mathbb{R}} x^k \psi_M(x)\mathrm{d}x = 0, \; k = 0, \dots, M-1.$$

This family of wavelets is called *Coiflets* after Roland Coifmann who inspired the construction by Daubechies.