Karlsruhe Institute of Technology

Department of Mathematics
Institute for Applied and Numerical Mathematics

Geometric Reconstruction in Bioluminescence Tomography

A Mumford-Shah like Approach for Finding the Support and Intensity of a Photon Source Inside an Organism

Tim Kreutzmann (joint work with Andreas Rieder)

1. Introduction

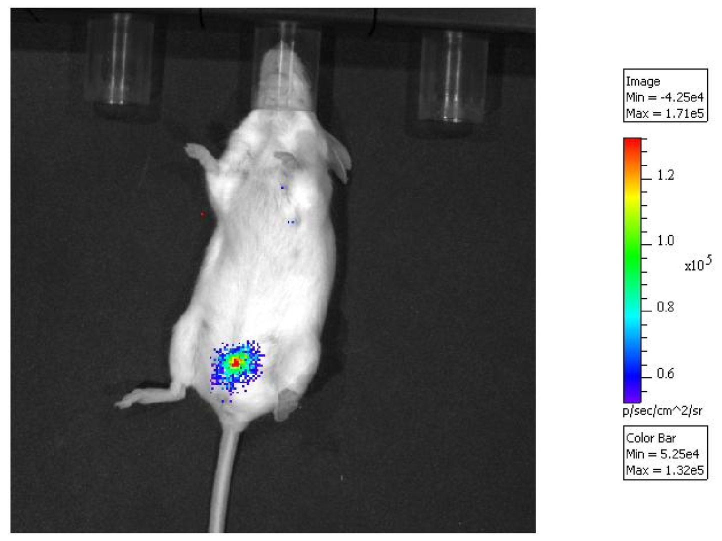

Figure 1: *Bioluminescence Image. Kindly provided by the group of Prof. Sahin at University Medical Center, Mainz.*

- Mathematical Model: The propagation of light in tissue can be modeled by the diffusion equation

$$-\mathrm{div}(D\nabla u) + \mu u = q \quad \text{in } \Omega, \qquad 2D\frac{\partial u}{\partial \nu} + u = g^- \quad \text{on } \partial\Omega \tag{1}$$

which is an approximation to the radiative transfer equation. The measurements are described by the boundary condition

$$D\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega.$$

- BLT Problem: Introducing the forward operator A mapping the source q to the Neumann values of the solution u of (1), the general BLT problem is to solve the operator equation

$$Aq = g.$$

- *A priori* Knowledge: We assume that

$$q = \lambda\chi_G, \quad \lambda \in [\lambda_{\min}, \lambda_{\max}],\ G \subset \Omega.$$

Under this assumption the BLT problem consists of solving the nonlinear operator equation

$$F(\lambda, G) := \lambda A\chi_G = g.$$

- Regularized Problem: The total variation of χ_G, which coincides with the perimeter of G, is considered as penalty term. This finally leads to the minimization of the Mumford-Shah like functional

$$J_\alpha(\lambda, G) := \frac{1}{2}\|F(\lambda, G) - g\|^2_{L^2} + \alpha \mathrm{Per}(G)$$

over $\Lambda \times \mathcal{L}$ with $\Lambda = [\lambda_{\min}, \lambda_{\max}]$ and $\mathcal{L}$ the set of all measurable subsets of Ω, i.e. to the problem

$$\text{Minimize } J_\alpha(\lambda, G) \text{ over } \Lambda \times \mathcal{L}. \tag{2}$$

2. Analysis of the Minimization Problem

- Existence of a Minimizer:

Theorem 2.1. *For all $\alpha > 0$ and $g \in L^2(\partial\Omega)$ there exists a solution $(\lambda^*, G^*) \in \Lambda \times \mathcal{L}$ of the problem (2), i.e.*

$$J_\alpha(\lambda^*, G^*) \le J_\alpha(\lambda, G) \qquad \text{for all } (\lambda, G) \in \Lambda \times \mathcal{L}.$$

- Stability:

Theorem 2.2. *Let $g_n \to g$ in L^2 as $n \to \infty$ and (λ^n, G^n) minimize*

$$J^n_\alpha(\lambda, G) := \frac{1}{2}\|F(\lambda, G) - g_n\|^2_{L^2} + \alpha\mathrm{Per}(G) \quad \text{over } \Lambda \times \mathcal{L}.$$

Then there exists a subsequence $\{(\lambda^{n_k}, G^{n_k})\}_k$ converging to a minimizer (λ^*, G^*) of J_α in the sense that

$$\|\lambda^{n_k}\chi_{G^{n_k}} - \lambda^*\chi_{G^*}\|_{L^2} \to 0 \quad \text{as } k \to \infty.$$

Furthermore, every convergent subsequence of $\{(\lambda^n, G^n)\}_n$ converges to a minimizer of J_α.

- Regularization Property:

Theorem 2.3. *Let g be in the range of F and choose the regularization parameter according to $\delta \mapsto \alpha(\delta)$ where*

$$\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta^2}{\alpha(\delta)} \to 0 \quad \text{as } \delta \to 0.$$

In addition, let $\{\delta_n\}_n$ be a positive null sequence and $\{g_n\}_n$ such that

$$\|g_n - g\|_{L^2} \le \delta_n.$$

Then, with the notation of Theorem 2.2, the sequence $\{(\lambda^n, G^n)\}$ of minimizers of $J^n_{\alpha(\delta_n)}$ possesses a subsequence converging to (λ^+, G^+) which satisfies

$$G^+ = \arg\min\{\mathrm{Per}(G) \mid G \in \mathcal{L}_g\}, \qquad \lambda^+ \in \{\lambda \in \Lambda \mid F(\lambda, G^+) = g\} \tag{3}$$

with $\mathcal{L}_g = \{G \in \mathcal{L} \mid \exists \lambda \in \Lambda \text{ with } F(\lambda, G) = g\}$. Furthermore, every convergent subsequence of $\{(\lambda^n, G^n)\}_n$ converges to a pair $(\lambda^\dagger, G^\dagger)$ with property (3).

3. Existence of Smooth Almost Stationary Points

- Approximate Variational Principle:

Theorem 3.1. *Let (λ^*, G^*) be a minimizer of J_α and λ^* an inner point of Λ. In the three-dimensional case, assume that G^* is a finite union of disjoint connected domains. Then for any $\varepsilon > 0$ sufficiently small we can find an intensity $\lambda^\varepsilon \in \Lambda$ and a C^2-domain G^ε satisfying*

$$J_\alpha(\lambda^\varepsilon, G^\varepsilon) - J_\alpha(\lambda^*, G^*) \le \varepsilon,$$
$$\|\lambda^\varepsilon\chi_{G^\varepsilon} - \lambda^*\chi_{G^*}\|_{L^1} \le \varepsilon,$$
$$\|J'_\alpha(\lambda^\varepsilon, G^\varepsilon)\|_{\mathbb{R}\times C^2 \to \mathbb{R}} \le \varepsilon.$$

Herein, $\partial_G J_\alpha(\lambda, \cdot)$ denotes the domain derivative of $J_\alpha(\lambda, \cdot)$.

4. Numerical Experiments

- Restriction to Star-shaped Domains: For the numerical experiments we only consider star-shaped domains and work on the linear space of parametrizations. All previous results hold true in this setting.
- Implementation: The discussed approach is implemented using trigonometric polynomials as parametrization of the domain and a projected gradient method for the minimization.

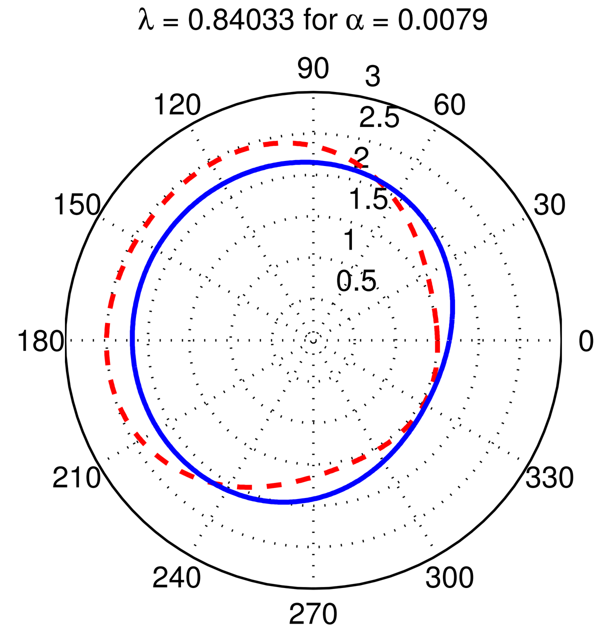

Figure 2: *Reconstruction (blue solid) and original source (red dashed) with $\alpha = 0.0079$ after 37 gradient iterations.*

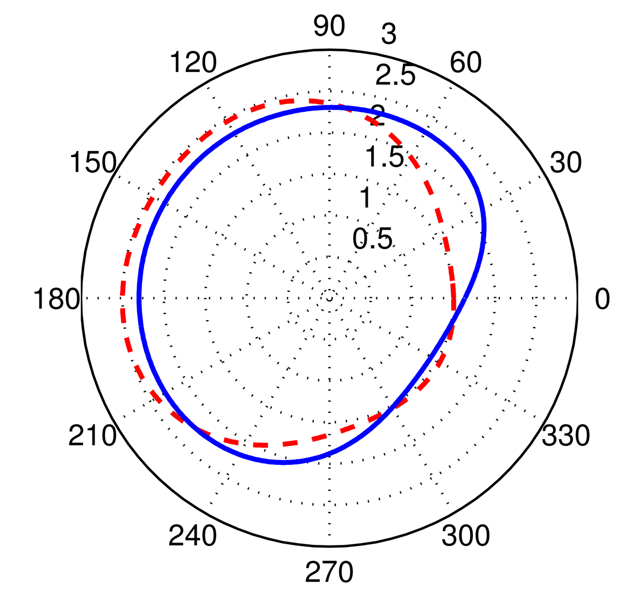

Figure 3: *Reconstruction (blue solid) and original source (red dashed) with 3% noise level and $\alpha = 0.008$ after 24 gradient iterations.*

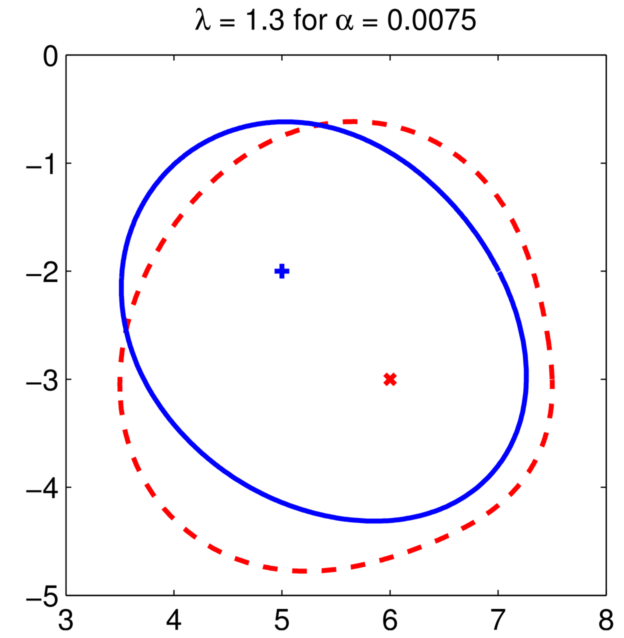

Figure 4: *Reconstruction (blue solid) and original source (red dashed) with $\alpha = 0.0075$ after 436 gradient iterations assuming a different midpoint.*

References

[1] W. Han, W. Cong and G. Wang, *Mathematical theory and numerical analysis of bioluminescence tomography*, Inverse Problems **22** (2006), pp. 1659–1675.

[2] T. Kreutzmann and A. Rieder, *Geometric reconstruction in bioluminescence tomography*, Preprint (2012).

[3] R. Ramlau and W. Ring, *Regularization of ill-posed Mumford-Shah models with perimeter penalization*, Inverse Problems **26** (2010), 115001.