A Numerical Solution Method for an Infinitesimal Elasto-Plastic Cosserat Model

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A micro-polar extension to infinitesimal elasticity

- We present a geometrically linear generalized continua of Cosserat micro-polar type in the elasto-plastic case.
- Starting with linear elasticity we postulate independent infinitesimal micro-rotations of the material. Thus, as a consequence of balance of angular momentum, stresses $\sigma$ are not symmetric any more.
- Cosserat regularize the mesh size dependence of localization computations where shear failure mechanisms play a dominant role.
- Models are of engineering interest: Diebels/Ehlers, Iordache/William, Dietsche/Steinmann/Willam, Ristinmaa/Vecci, de Borst, ...
- We restrict Cosserat micro-rotations to the elastic response of the material. Inelasticity is concerned as in Prandtl-Reuß plasticity. Thus, the elasto-plastic Cosserat problem is well-posed (Neff/Chełmiński).
Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be the reference configuration, and let $\Gamma_D \cup \Gamma_N = \partial \Omega$ be a decomposition of the boundary. We fix a time interval $[0, T]$.

Given data:

- displacement vector $\mathbf{u}_D : \Gamma_D \times [0, T] \rightarrow \mathbb{R}^d$,
- suitable infinitesimal micro-rotations $\bar{\mathbf{A}}_D : \Gamma_D \times [0, T] \rightarrow \mathfrak{so}(d)$,

where $\mathfrak{so}(d) = \{ \mathbf{\tau} \in \mathbb{R}^{d,d} : \mathbf{\tau}^T = -\mathbf{\tau} \}$ is the Lie algebra of skew-symmetric matrices, and a load functional

$$\ell(t, \mathbf{v}) = \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} \, d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N(t) \cdot \mathbf{v} \, d\mathbf{a}$$

depending on traction force densities $\mathbf{t}_N$ and body force densities $\mathbf{b}$.

The material is described by a linear elastic response depending on the Lamé constants $\lambda, \mu > 0$ and Cosserat constants $L_c > 0$ and $\mu_c \geq 0$. 

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Infinitesimal Elasto-Plastic Cosserat Model - Flow rule

Inelastic material behavior is modeled by a convex function

$$\phi: \text{Sym}(d) \rightarrow \mathbb{R}$$

determining the convex set of admissible stresses $K = \{\tau \in \text{Sym}(d): \phi(\tau) \leq 0\}$. We assume that $\phi$ is smooth for $\tau \neq 0$, and we assume $\phi(0) < 0$. In a first approach we choose the von Mises yield criterion $\phi(\tau) = |\text{dev}(\tau)| - K_0$ for a given constant $K_0 > 0$. Where we used $\text{dev}: \text{Sym}(d) \rightarrow \text{Sym}(d)$ with $\text{dev} \tau = \tau - \frac{1}{d} \text{tr} \tau$.

We have $P_K(\theta) = \theta - \max \{0, \gamma\} \frac{\text{dev}(\theta)}{|\text{dev}(\theta)|}$.

We use the following realization $C(\theta) \in \partial P_K(\theta)$, where $P_K(\theta)$ is the multi-valued derivative of the projection defined by $C(\theta) = \text{id}$ for $|\text{dev}(\theta)| \leq K_0$ and

$$C(\theta) = \frac{1}{d} \mathbf{I} \otimes \mathbf{I} + \frac{K_0}{|\text{dev}(\theta)|} \left( (\text{id} - \frac{1}{d} \mathbf{I} \otimes \mathbf{I}) - \frac{\text{dev}(\theta)}{|\text{dev}(\theta)|} \otimes \frac{\text{dev}(\theta)}{|\text{dev}(\theta)|} \right) \text{ for } |\text{dev}(\theta)| > K_0.$$
Infinitesimal Elasto-Plastic Cosserat Model - Equations

We want to determine displacements

\[
\mathbf{u} : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}^d,
\]

infinitesimal micro-rotations

\[
\bar{\mathbf{A}} : \Omega \times [0, T] \rightarrow \mathfrak{so}(d),
\]

non-symmetric stresses

\[
\mathbf{\sigma} : \Omega \times [0, T] \rightarrow \mathbb{R}^{d,d},
\]

symmetric plastic strains

\[
\mathbf{\varepsilon}_p : \Omega \times [0, T] \rightarrow \text{Sym}(d) \text{ with } \mathbf{\varepsilon}_p(0) = \mathbf{0},
\]

and a plastic multiplier

\[
\Lambda : \Omega \times [0, T] \rightarrow \mathbb{R},
\]

satisfying the essential boundary conditions and the equilibrium equations

\[
\begin{align*}
-\text{div} \mathbf{\sigma}(\mathbf{x}, t) &= \mathbf{b}(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times [0, T],
\mathbf{\sigma}(\mathbf{x}, t)\mathbf{n}(\mathbf{x}) &= \mathbf{t}_N(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Gamma_N \times [0, T],
-\mu L_c^2 \Delta \bar{\mathbf{A}}(\mathbf{x}, t) &= \mu_c \left( \text{skew}(\mathbf{D}\mathbf{u}(\mathbf{x}, t)) - \bar{\mathbf{A}}(\mathbf{x}, t) \right), \quad (\mathbf{x}, t) \in \Omega \times [0, T],
\mathbf{D}\bar{\mathbf{A}}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) &= \mathbf{0}, \quad (\mathbf{x}, t) \in \Gamma_N \times [0, T],
\end{align*}
\]

the constitutive relation

\[
\begin{align*}
\mathbf{\sigma}(\mathbf{x}, t) &= 2\mu \left( \text{sym}(\mathbf{D}\mathbf{u}(\mathbf{x}, t)) - \mathbf{\varepsilon}_p(\mathbf{x}, t) \right) + \lambda \text{div}(\mathbf{u})(\mathbf{x}, t) \mathbf{l} \\
&+ 2\mu_c \left( \text{skew}(\mathbf{D}\mathbf{u}(\mathbf{x}, t)) - \bar{\mathbf{A}}(\mathbf{x}, t) \right), \quad (\mathbf{x}, t) \in \Omega \times [0, T],
\end{align*}
\]
Infinitesimal Elasto-Plastic Cosserat Model - Equations

the complementary conditions for the yield criterion

\[ \Lambda(x, t) \phi(T_E(x, t)) = 0, \quad \Lambda(x, t) \geq 0, \quad \phi(T_E(x, t)) \leq 0, \quad (x, t) \in \Omega \times [0, T]. \]

and the flow rule

\[ \frac{d}{dt} \varepsilon_p(x, t) = \Lambda(x, t)D\phi(T_E(x, t)), \quad (x, t) \in \Omega \times [0, T], \]

depending on \( T_E(x, t) = 2\mu(\text{sym}(Du(x, t)) - \varepsilon_p(x, t)) \).

For given material history \( \varepsilon_p(t) \) at fixed time \( t \), the displacement and the micro-rotations are determined by minimizing the total energy

\[ I(u, A, \varepsilon_p) = E(\varepsilon(u), A, \varepsilon_p) - \ell(t, u), \]

with

\[ E(\varepsilon, A, \varepsilon_p) = \mu \int_{\Omega} |\text{sym}(\varepsilon) - \varepsilon_p|^2 \, dx + \frac{\lambda}{2} \int_{\Omega} \text{tr}(\varepsilon)^2 \, dx \]

\[ + \mu_c \int_{\Omega} |\text{skew}(\varepsilon) - A|^2 \, dx + \mu L_c^2 \int_{\Omega} |D\bar{A}|^2 \, dx. \]
Discretization in space

Let $h$ be a mesh size parameter, and let $\mathbf{V}_h \subset C^{0,1}(\Omega, \mathbb{R}^d)$ and $\mathcal{W}_h \subset C^{0,1}(\Omega, \mathfrak{s}\mathfrak{o}(d))$ be finite element spaces, and set

$$V_h(u_D) = \{ v \in V_h : v(x) = u_D(x) \text{ for } x \in D_h \},$$

$$W_h(A_D) = \{ B \in W_h : B(x) = A_D(x) \text{ for } x \in D'_h \}$$

where $D_h, D'_h \subset \Gamma_D$ are the sets of all nodal points on $\Gamma_D$ for $u$ and $A$.

Let $\Xi_h \subset \Omega$ be quadrature points and let $\omega_\xi$ be corresponding quadrature weights such that

$$\int_{\Omega} v \cdot w \, dx = \sum_{\xi \in \Xi_h} \omega_\xi \, v(\xi) \cdot w(\xi), \quad v, w \in V_h.$$ 

We set

$$\Lambda = \{ \Lambda : \Xi_h \to \mathbb{R} \},$$

$$\Sigma_h = \{ \tau : \Xi_h \to \mathbb{R}^{d,d} \},$$

and

$$E_h^p = \{ \tau : \Xi_h \to \mathfrak{sl}(d) \cap \text{Sym}(d) \},$$

where $\mathfrak{sl}(d) = \{ \tau \in \mathbb{R}^{d,d} : \text{tr}(\tau) = 0 \}$ is the Lie algebra of trace-free matrices.
Discretization in space

Determine

- displacements \( \mathbf{u} : [0, T] \rightarrow \mathbf{V}_h, \)
- stresses \( \sigma : [0, T] \rightarrow \Sigma_h, \)
- micro-rotations \( \bar{A} : [0, T] \rightarrow \mathcal{W}_h, \)
- plastic strains \( \varepsilon_p : [0, T] \rightarrow \mathbf{E}_h^p, \)
and a plastic multiplier \( \Lambda : [0, T] \rightarrow \Lambda \)

satisfying

- the equilibrium equations,
  \[
  \int_\Omega \sigma : \mathbf{Dv} \, d\mathbf{x} = \ell(\cdot, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h(0),
  \]
  \[
  \mu L_c^2 \int_\Omega D\bar{A} \cdot D\bar{B} \, d\mathbf{x} = \mu_c \int_\Omega (\text{skew}(D\mathbf{u}) - \bar{A}) : \bar{B} \, d\mathbf{x}, \quad \bar{B} \in \mathcal{W}_h(0)
  \]

- the essential boundary conditions,
- the constitutive relation,
- the complementary conditions (Kuhn-Tucker),
- and the flow rule.
Discretization in time

The model of incremental infinitesimal plasticity is obtained by a decomposition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval and backward Euler scheme. For $n = 1, 2, 3, \ldots$ the next increment depends on the material history described by $\varepsilon_p^{n-1}$, the new load $\ell^n[v] = \ell(t_n, v)$, and the new Dirichlet boundary values $u^n_D = u_D(t_n)$ and $\bar{A}^n_D = \bar{A}_D(t_n)$.

We compute the displacement vector $u^n \in V_h(u^n_D)$, the stresses $\sigma^n \in \Sigma_h$, the micro-rotations $\bar{A} \in W_h(\bar{A}^n_D)$, the plastic strains $\varepsilon_p^n \in E_p^h$, and the plastic multiplier $\Lambda^n \in \Lambda$ satisfying additionally the discrete flow-rule:

$$
\frac{1}{t_n - t_{n-1}} \left( \varepsilon_p^n(\xi) - \varepsilon_p^{n-1}(\xi) \right) = \Lambda^n(\xi) D\phi(T^n_E(\xi)) , \quad \xi \in \Xi_h ,
$$

depending on $T^n_E(\xi) = 2\mu \left( \text{sym}(Du^n(\xi)) - \varepsilon_p^n(\xi) \right)$.

Since the problem is rate-independent, we define $\gamma^n = (t_n - t_{n-1})\Lambda^n \in \Lambda$. 
Fully discrete elasto-plastic problem

Together, we can state the fully discrete elasto-plastic Cosserat problem.

For given \( \varepsilon_p^{n-1} \in \mathbf{E}_h^p \) find \( \sigma^n, T^n_E \in \Sigma_h, \ u^n \in \mathbf{V}_h(u^n_D), \ \bar{A} \in \mathcal{W}_h(\bar{A}_D^n) \) and \( \gamma^n \in \Lambda \) such that

\[
T^n_E(\xi) = 2\mu \left( \text{sym}(Du^n(\xi)) - \varepsilon_p^{n-1}(\xi) - \gamma^n(\xi)D\phi(T^n_E(\xi)) \right), \quad \xi \in \Xi_h,
\]

\[
\phi(T^n_E(\xi)) \leq 0, \quad \gamma^n(\xi)\phi(T^n_E(\xi)) = 0, \quad \gamma^n(\xi) \geq 0, \quad \xi \in \Xi_h,
\]

\[
\sigma^n(\xi) = T^n_E(\xi) + \lambda \text{div}(u^n)(\xi)I + 2\mu_c(\text{skew}(Du^n(\xi)) - \bar{A}^n(\xi)), \quad \xi \in \Xi_h,
\]

\[
\int_{\Omega} \sigma^n : D\mathbf{v} \, d\mathbf{x} = \ell^n[\mathbf{v}], \quad \mathbf{v} \in \mathbf{V}_h(0),
\]

\[
\mu L_c^2 \int_{\Omega} D\bar{A}^n \cdot D\bar{B} \, d\mathbf{x} = \mu_c \int_{\Omega} (\text{skew}(Du^n) - \bar{A}^n) : \bar{B} \, d\mathbf{x}, \quad \bar{B} \in \mathcal{W}_h(0).
\]
Lemma:
The fully discrete elasto-plastic problem is equivalent to the following nonlinear variational problem. For given $\varepsilon_p^{n-1}$ find $(u^n, \bar{A}^n) \in V_h(u^n_D) \times W_h(\bar{A}^n_D)$ such that

\[
\int_{\Omega} P_K \left( 2\mu (\text{sym} (D u^n) - \varepsilon_p^{n-1}) \right) : Dv \, dx + \lambda \int_{\Omega} \text{div}(u^n) \text{div}(v) \, dx
\]

\[
+ 2\mu_c \int_{\Omega} (\text{skew} (D u^n) - \bar{A}^n) : Dv \, dx = \ell^n[v], \quad v \in V_h(0),
\]

\[
\mu L_c^2 \int_{\Omega} D\bar{A}^n : D\bar{B} \, dx = \mu_c \int_{\Omega} (\text{skew} (D u^n) - \bar{A}^n) : \bar{B} \, dx, \quad \bar{B} \in W_h(0).
\]
Variational Formulation of the discrete problem

**Lemma:**
Any minimizer \((u^n, \bar{A}^n) \in \mathbf{V}_h(u_D^n) \times \mathcal{W}_h(\bar{A}_D^n)\) of the functional

\[
\mathcal{I}_{\text{incr}}^n(u, \bar{A}) = \mathcal{E}_{\text{incr}}(\varepsilon(u), \bar{A}, \varepsilon_p^{n-1}) - \ell_n[u]
\]
solves the nonlinear variational problem. Here \(\mathcal{E}_{\text{incr}}\) denotes the free energy of the incremental problem defined by

\[
\mathcal{E}_{\text{incr}}(Du, \bar{A}, \varepsilon_p) = \frac{1}{2\mu} \int_\Omega \psi_K(2\mu(\text{sym}(Du) - \varepsilon_p)) \, dx + \frac{\lambda}{2} \int_\Omega \text{tr}(Du)^2 \, dx
\]
\[
+ \mu_c \int_\Omega |\text{skew}(Du) - \bar{A}|^2 \, dx + \mu L_c^2 \int_\Omega |D\bar{A}|^2 \, dx.
\]

and a convex, non-negative, potential

\[
\psi_K(\theta) = \frac{1}{2} |\theta|^2 - \frac{1}{2} |\theta - P_K(\theta)|^2.
\]

for \(\theta \in \text{Sym}(d)\).
Numerical Solution Algorithm

We formulate a semi-smooth Newton method for the nonlinear variational problem

\[
(u^n, \tilde{A}^n) \in V_h(u^n_D) \times W_h(\tilde{A}_D^n): \quad F^n(u^n, \tilde{A}^n) = 0
\]

in every time step \( n \), where \( F^n \) is the first variation of \( I_{\text{incr}}^n \) defined by

\[
F^n(u, \tilde{A})[v, \tilde{B}] = D I_{\text{incr}}^n(u, \tilde{A})[v, \tilde{B}], \quad (v, \tilde{B}) \in V_h(0) \times W_h(0).
\]

The functional \( F^n \) is semi-smooth, and the second variation \( \partial^2 I_{\text{incr}}^n = \partial F^n \) is multi-valued. Thus, the corresponding semi-smooth Newton method can be formally written as

\[
0 \in F^n(u^{n,k}, \tilde{A}^{n,k}) + \partial F^n(u^{n,k}, \tilde{A}^{n,k})[u^{n,k+1} - u^{n,k}, \tilde{A}^{n,k+1} - \tilde{A}^{n,k}].
\]

We consider the special case of the von Mises flow rule. The Newton increment is realized using the consistent linearization \( C(\theta) \).
Benchmark problem for parameter study of $\mu_c$

Let $\Omega = (0, 10) \times (0, 10) \setminus B_1(10, 0)$. We use Q1 discretization and present results for 198147 unknowns on uniform refinement level 4. We have chosen the parameters $K_0 = 450$, $\lambda = 110743.8$, $\mu = 80193.8$ and $L_c = 0.020833$. And apply traction force by Neumann boundary condition according to:

$$\ell(t, v) = 100 t \int_0^{10} v(x_1, 10) dx_1.$$
Numerical Experiment with M++

Cosserat Model \( (\mu_c = \mu) \): Effective plastic strain

Prandtl-Reuß \( (\mu_c = 0) \): Effective plastic strain

\[
\mu = \mu_c
\]

\[
\mu_c = 0
\]

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\mu_c = \mu
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\mu = \mu_c
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\mu_c = 0
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\[
\mu = \mu_c
\]
Numerical Experiment with M++

Load-displacement curve: displacement $u$ is evaluated at special point $z_0$. 

- $\mu_c = \mu$
- $\mu_c = 0.01\mu$
- $\mu_c = 0.001\mu$
- $\mu_c = 0.0005\mu$
- $\mu_c = 0.0001\mu$
- $\mu_c = 0$
Summary and Outlook

- The Elasto-Plastic Cosserat Model with pure Dirichlet data is well-defined. Solution exists global in time.
- Elasto-Plastic Cosserat Model is a regularization for classical plasticity.
- Complete finite element analysis (dependent on $\mu_c$) is available and will be published.
- Future work will be the analysis and implementation of nonlinear Cosserat Models.