Tangential Cone Condition and other open problems in Electrical Impedance Tomography

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The Tangential Cone Condition (TCC)

EIT: Continuous Model

EIT: Complete Electrode Model

Conclusion
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Iterative regularizations for nonlinear problems

\[ F : D(F) \subset X \to Y, \quad X, Y \text{ Hilbert spaces} \]

\[ F(x) = y^\delta \]

Iterative schemes:

\[ x_{n+1} = x_n + s_n, \quad x_0 \in D(F), \text{ where } s_n \text{ is determined} \]

from the linearization \( A_n s = b_n^\delta \) about \( x_n \).

\[ (A_n = F'(x_n), \quad b_n^\delta = y^\delta - F(x_n)) \]

Examples:

- nonlinear Landweber (Hanke, Neubauer, Scherzer '95)
- nonlinear gradient decent (Scherzer '96)
- iteratively regularized Gauss-Newton methods (Bakushinsky '92, Blaschke (Kaltenbacher), Neubauer & Scherzer '97, Kaltenbacher '98, ...)
- Newton-CG (Hanke '98, R. '05, Lechleiter & R. '07)
The need for the tangential cone condition

Bound of linearization error given by Taylor expansion

\[ \| F(v) - F(u) - F'(u)(v - u) \| \lesssim \| v - u \|^2 \]

offers only little control for ill-posed problems.
The need for the tangential cone condition

Bound of linearization error given by Taylor expansion

$$\|F(v) - F(u) - F'(u)(v - u)\| \lesssim \|v - u\|^2$$

offers only little control for ill-posed problems.

Let $F$ be completely continuous and let $\{u_n\}$ converge weakly to $u$ where $\|u - u_n\| = c$ for all $n$. Then,

$$\|F(u_n) - F(u) - F'(u)(u_n - u)\| \lesssim \|u_n - u\|^2 \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$= \text{const as } n \rightarrow \infty$$
F satisfies the *tangential cone condition* (Scherzer ’93) locally about $x^+ \in D(F)$ if there is a positive constant $\omega < 1$ such that

$$
\|F(v) - F(u) - F'(u)(v - u)\| \leq \omega \|F(v) - F(u)\|, \quad v, u \in B_\rho(x^+)
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F satisfies the \textit{tangential cone condition} (Scherzer '93) locally about $x^+ \in D(F)$ if there is a positive constant \(\omega < 1\) such that

$$\|F(v) - F(u) - F'(u)(v - u)\| \leq \omega \|F(v) - F(u)\|, \quad v, u \in B_\rho(x^+)$$

\textbf{Lemma} If TCC then $N(F'(v)) = N(F'(u))$ for all $v, u \in B_\rho(x^+)$. Moreover, if $u - v \in N(F'(x^+))$ then $F(u) = F(v)$. Especially: $F'(x^+)$ is injective whenever $F$ is injective.
Tangential cone condition (TCC)

\( F \) satisfies the \textit{tangential cone condition} (Scherzer '93) locally about \( x^+ \in D(F) \) if there is a positive constant \( \omega < 1 \) such that

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\| F(v) - F(u) - F'(u)(v-u) \| \leq \omega \| F(v) - F(u) \|, \quad v, u \in B_\rho(x^+)
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\textbf{Lemma} If TCC then \( N(F'(v)) = N(F'(u)) \) for all \( v, u \in B_\rho(x^+) \).
Moreover, if \( u - v \in N(F'(x^+)) \) then \( F(u) = F(v) \).
Especially: \( F'(x^+) \) is injective whenever \( F \) is injective.

\textbf{Theorem} (Hofmann & Scherzer '98) If TCC about \( x^+ \) then:
The nonlinear problem is locally ill-posed about \( x^+ \) iff \( F'(x^+) \) has a non-closed range in \( Y \).
EIT: Continuous Model
Given $f \in H_{-1/2}(\partial B)$ find $u \in H_1^1(B)$:

$$\text{div}(\sigma \nabla u) = 0 \text{ in } B, \quad \sigma \partial_n u = f \text{ on } \partial B$$
Given $f \in H^{-1/2}_\diamond (\partial B)$ find $u \in H^1_\diamond (B)$:

$$\int_B \sigma \nabla u \nabla v \, dx = \int_{\partial B} f v \, dS \quad \forall v \in H^1_\diamond (B)$$

$\Lambda_\sigma : f \mapsto u|_{\partial B}$ Neumann to Dirichlet map
Given \( f \in H_{\diamond}^{-1/2}(\partial B) \) find \( u \in H^1(B) \):

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\]

\( \Lambda_\sigma : f \mapsto u|_{\partial B} \) Neumann to Dirichlet map

EIT forward operator

\[
F : D(F) \subset L^\infty(B) \rightarrow \mathcal{L}(L^2_{\diamond}(\partial B)), \quad \sigma \mapsto \Lambda_\sigma,
\]

with \( D(F) = \{ \sigma \in L^\infty(B) : \sigma \geq \sigma_0 > 0 \} \).

In other words: \( F(\sigma)f = u|_{\partial B} \)

\( F \) is injective (Astala and Päivärinta, 2006)
Let $\sigma \in \text{int}(D(F))$. Then,

$$F'(\sigma) \in \mathcal{L}(L^\infty(B), \mathcal{L}(L^2_\diamond(\partial B)))$$

is given by

$$F'(\sigma)[h]f := w|_{\partial B} \in L^2_\diamond(\partial B)$$

where

$$\int_B \sigma \nabla w \nabla \varphi \, dx = - \int_B h \nabla (F(\sigma)f) \nabla \varphi \, dx \quad \forall \varphi \in H^1_\diamond(B).$$
On the injectivity of $F'(\sigma)$

As $F$ is injective on $D(F)$ TCC can only hold about conductivities $\sigma$ for which $F'(\sigma)$ is injective as well.

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There has been no progress since the pioneering work of Calderón in 1980:

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$$F'(\sigma)[h] = 0 \iff \int_B h \nabla u(\sigma, f) \cdot \nabla u(\sigma, g) \, dx = 0 \quad \forall f, g \in L^2_0(\partial B)$$
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$\sigma = 1$: harmonic functions are admissible potentials.
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\]

\( \sigma = 1 \): harmonic functions are admissible potentials.

For the harmonic functions \( u(f)(x) = \exp(ik \cdot x + \ell \cdot x) \) and \( u(g)(x) = \exp(ik \cdot x - \ell \cdot x) \) with \( k, \ell \in \mathbb{R}^2, |k| = |\ell|, k \cdot \ell = 0 \), we obtain

\[
0 = \int_B h \exp(i \, 2k \cdot x) \, dx \quad \forall k \in \mathbb{R}^2.
\]

Hence, \( h = 0 \).
On the injectivity of $F'(\sigma)$ (continued)

**Theorem** (Gebauer 07) Let $\Omega_1, \Omega_2 \subset B$ be open with $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$. Furthermore, let $B \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ be connected and $\overline{B} \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ contain a relatively open subset $S$ of $\partial B$. Then there exists a sequence of currents $\{f_n\} \subset L^2_\diamond(S)$ and corresponding potentials $\{u_n\}$, defined by the weak formulation of

$$\text{div}(\sigma \nabla u_n) = 0, \quad \sigma \partial_\nu u_n |_{\partial B} = \begin{cases} f_n & \text{on } S, \\ 0 & \text{otherwise}, \end{cases}$$

such that

$$\lim_{n \to \infty} \int_{\Omega_1} |\nabla u_n|^2 dx = \infty \text{ and } \lim_{n \to \infty} \int_{\Omega_2} |\nabla u_n|^2 dx = 0.$$
\[ \text{div}(\sigma \nabla u_n) = 0, \quad \sigma \partial_{\nu} u_n|_{\partial B} = \begin{cases} f_n & \text{on } S, \\ 0 & \text{otherwise}, \end{cases} \]
On the injectivity of $F'(\sigma)|_{H^{1+}(B)}$

Assume: $h \in H^{1+}(B)$ with $h|_{\partial B} \neq 0$.

$$\Rightarrow \exists \ U \subset B \text{ (open, connected)} \text{ with } \overline{U} \cap \partial B \neq \emptyset \text{ and } \text{sgn}(h)|_{\overline{U}} = \text{const} \neq 0.$$

Pick $S \subset \overline{U} \cap \partial B$ and an open ball $\Omega_1 \subset U$. Set $\Omega_2 = B \setminus \overline{U}$. 
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Assume: $h \in H^{1+}(B)$ with $h|_{\partial B} \neq 0$.

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Pick $S \subset \overline{U} \cap \partial B$ and an open ball $\Omega_1 \subset U$. Set $\Omega_2 = B \setminus \overline{U}$. 

\[
\Omega_1 \cap \Omega_2 = B \setminus U
\]
Assume: \( h \in H^{1+}(B) \) with \( h|_{\partial B} \neq 0 \).

\[ \Rightarrow \exists U \subset B \text{ (open, connected) with } \overline{U} \cap \partial B \neq \emptyset \text{ and } \text{sgn}(h)|_{\overline{U}} = \text{const} \neq 0. \]

Pick \( S \subset \overline{U} \cap \partial B \) and an open ball \( \Omega_1 \subset U \). Set \( \Omega_2 = B \setminus \overline{U} \).

By Gebauer's theorem

\[ \left| \int_B h \left| \nabla u(\sigma, f_n) \right|^2 \, dx \right| \xrightarrow{n \to \infty} \infty. \]
On the injectivity of $F'(\sigma)|_{H^{1+}(B)}$ 

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By Gebauer’s theorem

$$\left| \int_B h |\nabla u(\sigma, f_n)|^2 \, dx \right| \xrightarrow{n \to \infty} \infty.$$ 

Assume $h \in N(F'(\sigma))$. Then,

$$\int_B h |\nabla u(\sigma, f)|^2 \, dx = 0 \ \forall f \in L^2_\diamond(\partial B)$$

contradicting the above limit. Hence, $h \notin N(F'(\sigma))$.

Remainder: $F'(\sigma)[h] = 0 \iff \int_B h \nabla u(\sigma, f) \cdot \nabla u(\sigma, g) \, dx = 0 \ \forall f, g \in L^2_\diamond(\partial B)$
On the injectivity of $F'(\sigma)|_{H^{1+}(B)}$

Assume: $h \in H^{1+}(B)$ with $h|_{\partial B} \neq 0$.

$\implies \exists U \subset B$ (open, connected) with $\overline{U} \cap \partial B \neq \emptyset$ and $\text{sgn}(h)|_{\overline{U}} = \text{const} \neq 0$.

Pick $S \subset \overline{U} \cap \partial B$ and an open ball $\Omega_1 \subset U$. Set $\Omega_2 = B \setminus \overline{U}$.

By Gebauer's theorem

$$\left| \int_B h |\nabla u(\sigma, f_n)|^2 \, dx \right| \xrightarrow{n \to \infty} \infty.$$

Assume $h \in \mathcal{N}(F'(\sigma))$. Then,

$$\int_B h |\nabla u(\sigma, f)|^2 \, dx = 0 \ \forall f \in L^2_\diamond(\partial B)$$

contradicting the above limit. Hence, $h \notin \mathcal{N}(F'(\sigma))$.

Line of reasoning remains correct if $h|_{\partial B} = 0$ but $\text{sgn}(h)|_{\text{int}(U)} = \text{const} \neq 0$. 
On the injectivity of $F'(\sigma)|_{H^{1+}(B)}$ (continued)

$\mathsf{supph} \subseteq B$
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On the injectivity of $F'(\sigma)|_{H^{1+}(B)}$ (continued)

By Gebauer’s theorem

$$\left| \int_B h|\nabla u(\sigma, f_n)|^2 dx \right| \xrightarrow{n \to \infty} \infty.$$  

Hence, $h \notin N(F'(\sigma)).$
On the injectivity of $F'(\sigma)|_{H^{1+}(B)}$: summary

$$\mathcal{K}(\sigma) = \{ h \in H^{1+}(B) : h|_{\partial B} \neq 0 \}$$

$$\bigcup \{ h \in H^{1+}(B) : \exists U \subset \text{supph open, } \partial U \cap \partial \text{supph} \neq \emptyset, \text{ sgn}(h)|_{\text{int}(U)} = \text{const} \neq 0 \}$$
On the injectivity of $F'(\sigma)|_{H^{1+}(B)}$: summary

$$\mathcal{H}(\sigma) = \left\{ h \in H^{1+}(B) : h|_{\partial B} \neq 0 \right\}$$

$$\bigcup \left\{ h \in H^{1+}(B) : \exists U \subset \text{supp} h \text{ open}, \partial U \cap \partial \text{supp} h \neq \emptyset, \right.$$

$$\text{sgn}(h)|_{\text{int}(U)} = \text{const} \neq 0 \left. \right\}$$

**Theorem** $\mathcal{H}(\sigma) \cap N(F'(\sigma)) = \emptyset$
Unfortunately, there exist pathological $h$’s for which we cannot decide whether $F'(\sigma)[h] \neq 0$ or not.
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**Example** Let $B$ the circular disc with radius 1 centered about the origin. Consider $h : B \to \mathbb{R}$, $h(x) = r(|x|)$, where $r$ is sketched below:
Define a Finite Element space $V_\ell$ as follows:

$$V_\ell := R_B P_\ell E_D H^{1+}(B) \subset H^{1+}(B)$$

Observe: $V_\ell \cap N(F'(\sigma)) = \emptyset$

$$F: V^+_\ell \subset L^\infty(B) \to \mathcal{L}(L^2_\diamond(\partial B))$$

$$V^+_\ell := V_\ell \cap D(F) = \{\sigma_\ell \in V_\ell : \sigma_\ell \geq \sigma_0 > 0\}$$
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**Theorem** If $\sigma_\ell \in \text{int}(V_\ell^+)$ then there is a ball $U(\sigma_\ell) \subset V_\ell$ such that

$$\|F(\tau_\ell) - F(\gamma_\ell) - F'(\gamma_\ell)[\tau_\ell - \gamma_\ell]\|_{\mathcal{L}(L^2_\diamond(\partial B))}$$

$$\leq C_\ell \|\tau_\ell - \gamma_\ell\|_{H^{1+}(B)} \|F(\tau_\ell) - F(\gamma_\ell)\|_{\mathcal{L}(L^2_\diamond(\partial B))}$$

for any $\tau_\ell, \gamma_\ell \in U(\sigma_\ell)$ where

$$C_\ell \sim \sup \left\{ \frac{\|F'(\xi_\ell)[h_\ell]\|_{L^2_\diamond(\partial B) \to H^{1}(B)}}{\|F'(\xi_\ell)[h_\ell]\|_{L^2_\diamond(\partial B) \to L^2_\diamond(\partial B)}} : (\xi_\ell, h_\ell) \in U(\sigma_\ell) \times V_\ell \right\}.$$
EIT: Complete Electrode Model
\[ \text{current } I_i, \text{ voltage } U_k, \text{ electrodes } E_j, j = 1, \ldots, p \]

\[ \text{div}(\sigma \nabla u) = 0 \text{ in } B \]
\[ u + z_j \sigma \partial_n u = U_j \text{ on } E_j \]
\[ \sigma \partial_n u = 0 \text{ on } \partial B \setminus \bigcup_j E_j \]
\[ \frac{1}{|E_j|} \int_{E_j} \sigma \partial_n u \, dS = I_j = f|_{E_j} \]
Given $f \in \mathcal{E}_p := \{ j \in \text{span}\{\chi_{E_1}, \ldots, \chi_{E_p}\} : \int_{\partial B} j \, dS = 0 \} \subset L^2(\partial B)$ find $(u, U) \in H^1(B) \oplus \mathcal{E}_p$:

$$b_{\sigma}((u, U), (w, W)) = \int_{\partial B} fW \, dS \quad \forall (w, W) \in H^1(B) \oplus \mathcal{E}_p \quad (1)$$

where

$$b_{\sigma}((v, V), (w, W)) = \int_B \sigma \nabla v \cdot \nabla w \, dx + \sum_{j=1}^p \frac{1}{z_j} \int_{E_j} (v - V)(w - W) \, dS$$

(Existence & Uniqueness: Cheney, Isaacson & Somersalo, 1992)
The forward operator

\[ F_p : D(F) \subset L^\infty(B) \rightarrow \mathcal{L}(\mathcal{E}_p), \quad \sigma \mapsto \{ f \mapsto U \}, \quad F_p(\sigma)f = U, \]

where \( U \) is the second component of the solution of (1).
The forward operator

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Is \( F_p \) injective?
The forward operator

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Is \( F_p \) injective? Certainly not!
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How many independent measurements does CEM provide?
At most \((p - 1)^2\) because \(\dim \mathcal{E}_p = p - 1\).
The forward operator

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How many independent measurements does CEM provide? At most \((p - 1)^2\) because \( \dim \mathcal{E}_p = p - 1 \).

However,

\[ \langle F_p(\sigma)f, g \rangle_{L^2(\partial B)} = \langle f, F_p(\sigma)g \rangle_{L^2(\partial B)}. \]

Recall: \( \Lambda_\sigma \) is self-adjoint as well.
The forward operator

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Recall: \( \Lambda_\sigma \) is self-adjoint as well.

Thus, the DOF in \( F_p(\sigma) \) are the DOF of a symmetric matrix of order \( p - 1 \), namely \( p(p - 1)/2 \).
The forward operator

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Thus, the DOF in \( F_p(\sigma) \) are the DOF of a symmetric matrix of order \( p - 1 \), namely \( p(p - 1)/2 \).

Conjecture If \( V \subset L^\infty(B) \) with \( \dim V \leq p(p-1)/2 \) then
\[ F_p : D(F) \cap V \subset V \rightarrow \mathcal{L}(E_p) \] is injective.
As

\[ F_p : D(F') \cap V \subset V \rightarrow \mathcal{L}(\mathcal{E}_p) \]

is a mapping between finite dimensional spaces TCC holds locally about \( \sigma \in \text{int}(D(F') \cap V) \) whenever \( F'_p(\sigma) \in \mathcal{L}(V, \mathcal{L}(\mathcal{E}_p)) \) is injective.
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We have that
\[ F_p'(\sigma)[\eta] = 0 \iff \int_B \eta \nabla u(\sigma, f) \cdot \nabla u(\sigma, g) \, dx = 0 \quad \forall f, g \in \mathcal{E}_p \]
where \( u = u(\sigma, f) \in H^1(B) \) is the first component of the solution of
\[ b_\sigma((u, U), (w, W)) = \int_{\partial B} f W \, dS \quad \forall (w, W) \in H^1(B) \oplus \mathcal{E}_p. \]
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\[ b_\sigma((u, U), (w, W)) = \int_{\partial B} f W \, dS \quad \forall (w, W) \in H^1(B) \oplus \mathcal{E}_p. \]

**Conjecture** If \( V \subset L^\infty(B) \) with \( \dim V \leq p(p-1)/2 \) then

\[ F'_p(\sigma) \text{ is injective for any } \sigma \in \text{int}(D(F') \cap V). \]
Conclusion
What to remember from this talk

- TCC is a vital ingredient for the convergence analysis of iterative regularization schemes for nonlinear ill-posed problems.

- As the forward operator of the continuous model for EIT is injective, a necessary prerequisite for the TCC to hold is the injectivity of the Frechét derivative. We have shown that only 'pathological' elements can possibly be in the Null space. If we restrict the conductivities to a finite dimensional space, say, a finite element space, then TCC holds. Unfortunately, this is not an adequate setting for the continuous model.

- On the other hand, the CEM offers only finitely many independent measurements. Therefore, a finite dimensional setting is necessary to have injectivity of the forward operator. We conjectured that injectivity of the forward operator and its derivative hold if the number of DOF of the searched-for conductivity is at most the number of independent measurements. If the conjectures apply then TCC holds.
TCC is a vital ingredient for the convergence analysis of iterative regularization schemes for nonlinear ill-posed problems.

As the forward operator of the continuous model for EIT is injective, a necessary prerequisite for the TCC to hold is the injectivity of the Frechét derivative. We have shown that only 'pathological' elements can possibly be in the Null space. If we restrict the conductivities to a finite dimensional space, say, a finite element space, then TCC holds. Unfortunately, this is not an adequate setting for the continuous model.

On the other hand, the CEM offers only finitely many independent measurements. Therefore, a finite dimensional setting is necessary to have injectivity of the forward operator. We conjectured that injectivity of the forward operator and its derivative hold if the number of DOF of the searched-for conductivity is at most the number of independent measurements. If the conjectures apply then TCC holds.
TCC is a vital ingredient for the convergence analysis of iterative regularization schemes for nonlinear ill-posed problems.

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Thank you for your attention!