Tangential Cone Condition and other open problems in Electrical Impedance Tomography

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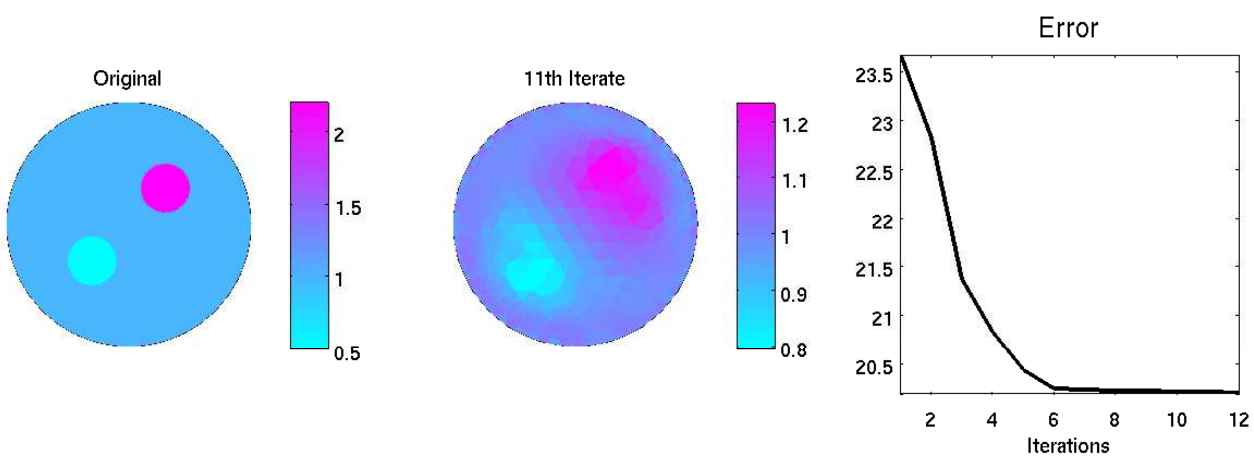

Overview

The Tangential Cone Condition (TCC)

EIT: Continuous Model

EIT: Complete Electrode Model

Conclusion

The Tangential Cone Condition (TCC)

Iterative regularizations for nonlinear problems

$$F : \mathsf{D}(F) \subset X \to Y, \quad X, Y \text{ Hilbert spaces}$$

$$F(x) = y^\delta$$

Iterative schemes: $x_{n+1} = x_n + s_n$, $x_0 \in \mathsf{D}(F)$, where s_n is determined from the linearization $A_n s = b_n^\delta$ about x_n.
($A_n = F'(x_n)$, $b_n^\delta = y^\delta - F(x_n)$)

Examples:

- nonlinear Landweber (Hanke, Neubauer, Scherzer '95)
- nonlinear gradient decent (Scherzer '96)
- iteratively regularized Gauss-Newton methods (Bakushinsky '92, Blaschke(Kaltenbacher), Neubauer & Scherzer '97, Kaltenbacher '98, ...)
- Newton-CG (Hanke '98, R. '05, Lechleiter & R. '07)

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The need for the tangential cone condition

Bound of linearization error given by Taylor expansion

$$\|F(v) - F(u) - F'(u)(v - u)\| \lesssim \|v - u\|^2$$

offers only little control for ill-posed problems.

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Let F be completely continuous and let $\{u_n\}$ converge weakly to u where $\|u - u_n\| = c$ for all n. Then,

$$\underbrace{\|F(u_n) - F(u) - F'(u)(u_n - u)\|}_{\to 0 \text{ as } n \to \infty} \lesssim \underbrace{\|u_n - u\|^2}_{= \text{const as } n \to \infty}$$

Tangential cone condition (TCC)

F satisfies the *tangential cone condition* (Scherzer '93) locally about $x^+ \in \mathrm{D}(F)$ if there is a positive constant $\omega < 1$ such that

$$\|F(v) - F(u) - F'(u)(v-u)\| \leq \omega \|F(v) - F(u)\|, \quad v, u \in B_\rho(x^+)$$

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Lemma If TCC then $\mathsf{N}(F'(v)) = \mathsf{N}(F'(u))$ for all $v, u \in B_\rho(x^+)$.
Moreover, if $u - v \in \mathsf{N}(F'(x^+))$ then $F(u) = F(v)$.
Especially: $F'(x^+)$ is injective whenever F is injective.

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Especially: $F'(x^+)$ is injective whenever F is injective.

Theorem (Hofmann & Scherzer '98) If TCC about x^+ then:
The nonlinear problem is locally ill-posed about x^+ iff $F'(x^+)$ has a non-closed range in Y.

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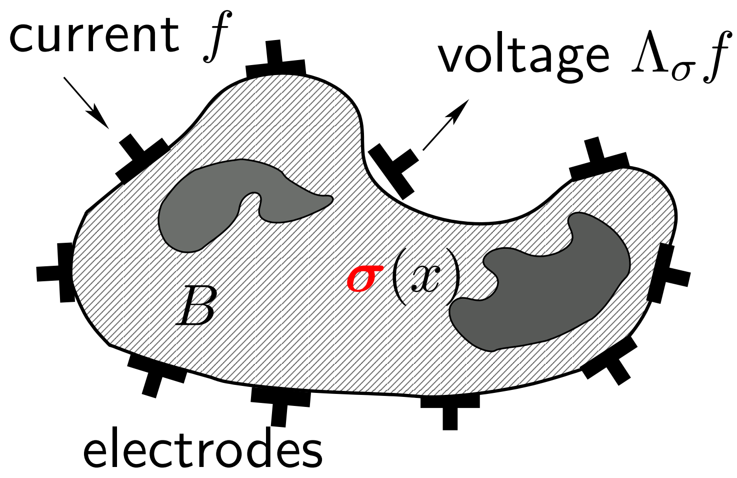
EIT: Continuous Model

Governing equation



Given $f \in H_\diamond^{-1/2}(\partial B)$ find $u \in H_\diamond^1(B)$:

$$\mathrm{div}(\sigma \nabla u) = 0 \text{ in } B, \quad \sigma \partial_{\mathbf{n}} u = f \text{ on } \partial B$$

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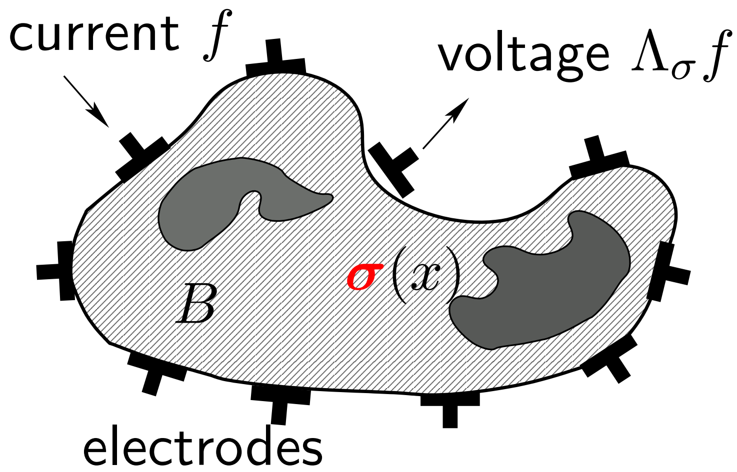

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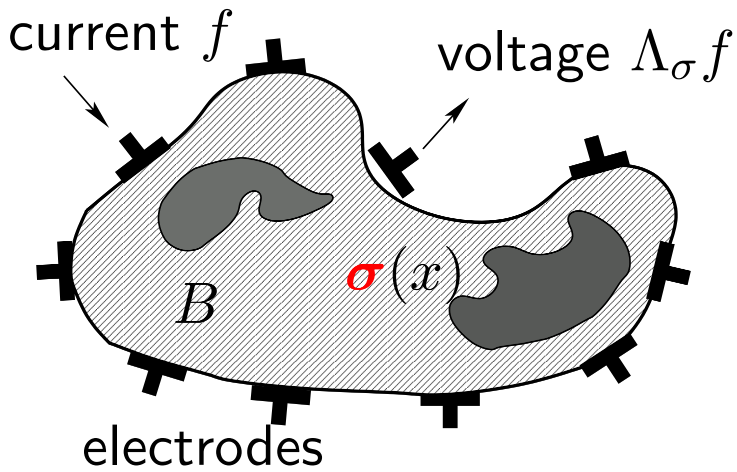
$$\int_B \sigma \nabla u \nabla v \, \mathrm{d}x = \int_{\partial B} f v \, \mathrm{d}S \quad \forall v \in H_\diamond^1(B)$$

$\Lambda_\sigma : f \mapsto u|_{\partial B}$ Neumann to Dirichlet map



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EIT forward operator

$$F : \mathrm{D}(F) \subset L^\infty(B) \to \mathfrak{L}\big(L_\diamond^2(\partial B)\big), \quad \sigma \mapsto \Lambda_\sigma,$$

with $\mathrm{D}(F) = \{\sigma \in L^\infty(B) : \sigma \geq \sigma_0 > 0\}$.

In other words: $F(\sigma)f = u|_{\partial B}$

F is injective (Astala and Päivärinta, 2006)

Frechét differentiability of EIT operator

Let $\sigma \in \text{int}(\mathsf{D}(F))$. Then,

$$F'(\sigma) \in \mathfrak{L}\big(L^\infty(B), \mathfrak{L}\big(L^2_\diamond(\partial B)\big)\big)$$

is given by

$$F'(\sigma)[h]f := w|_{\partial B} \in L^2_\diamond(\partial B)$$

where

$$\int_B \sigma \nabla w \nabla \varphi \, \mathrm{d}x = -\int_B h \nabla \big(F(\sigma) f\big) \nabla \varphi \, \mathrm{d}x \quad \forall \varphi \in H^1_\diamond(B).$$

On the injectivity of $F'(\sigma)$

As F is injective on $\mathsf{D}(F)$ TCC can only hold about conductivities σ for which $F'(\sigma)$ is injective as well.

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For the harmonic functions $u(f)(x) = \exp(\imath k \cdot x + \ell \cdot x)$ and $u(g)(x) = \exp(\imath k \cdot x - \ell \cdot x)$ with $k, \ell \in \mathbb{R}^2$, $|k| = |\ell|$, $k \cdot \ell = 0$, we obtain

$$0 = \int_B h \exp(\imath\, 2k \cdot x) \mathrm{d}x \quad \forall k \in \mathbb{R}^2.$$

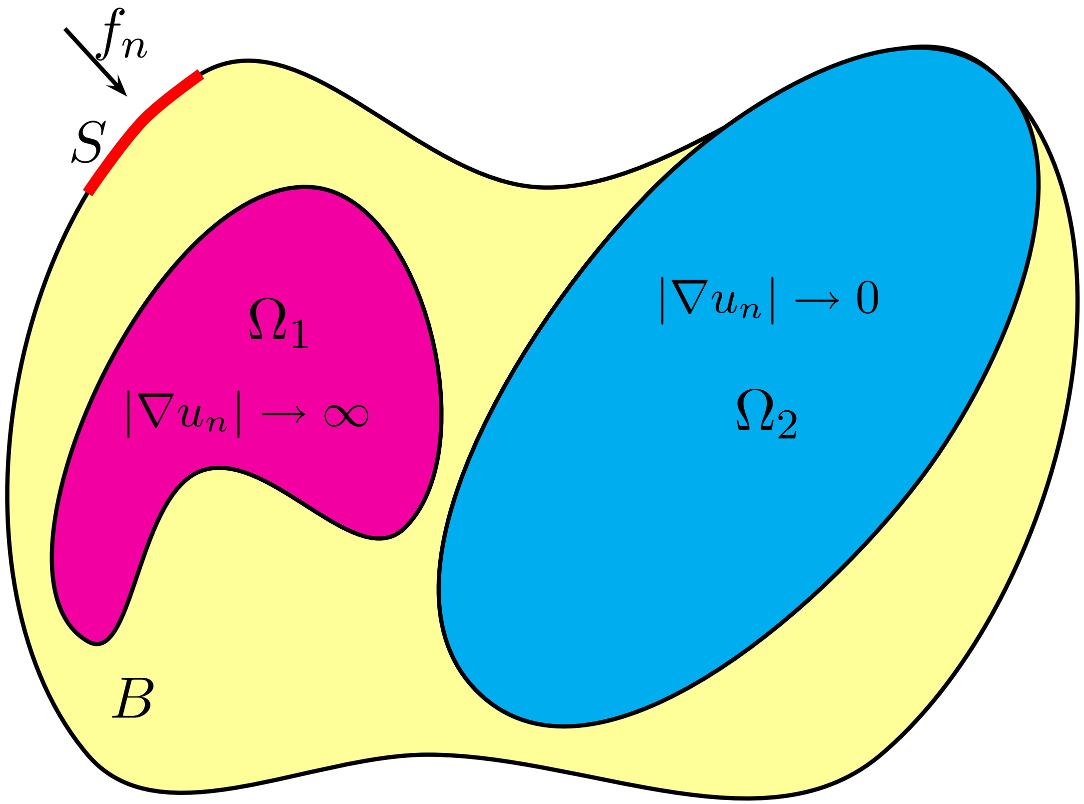
Hence, $h = 0$.

Theorem(Gebauer 07) Let Ω_1, $\Omega_2 \subset B$ be open with $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$. Furthermore, let $B \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ be connected and $\overline{B} \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ contain a relatively open subset S of ∂B. Then there exists a sequence of currents $\{f_n\} \subset L^2_\diamond(S)$ and corresponding potentials $\{u_n\}$, defined by the weak formulation of

$$\operatorname{div}\big(\sigma \nabla u_n\big) = 0, \quad \sigma \partial_\nu u_n|_{\partial B} = \begin{cases} f_n & \text{on } S, \\ 0 & \text{otherwise,} \end{cases}$$

such that

$$\lim_{n\to\infty} \int_{\Omega_1} |\nabla u_n|^2 \mathrm{d}x = \infty \text{ and } \lim_{n\to\infty} \int_{\Omega_2} |\nabla u_n|^2 \mathrm{d}x = 0.$$



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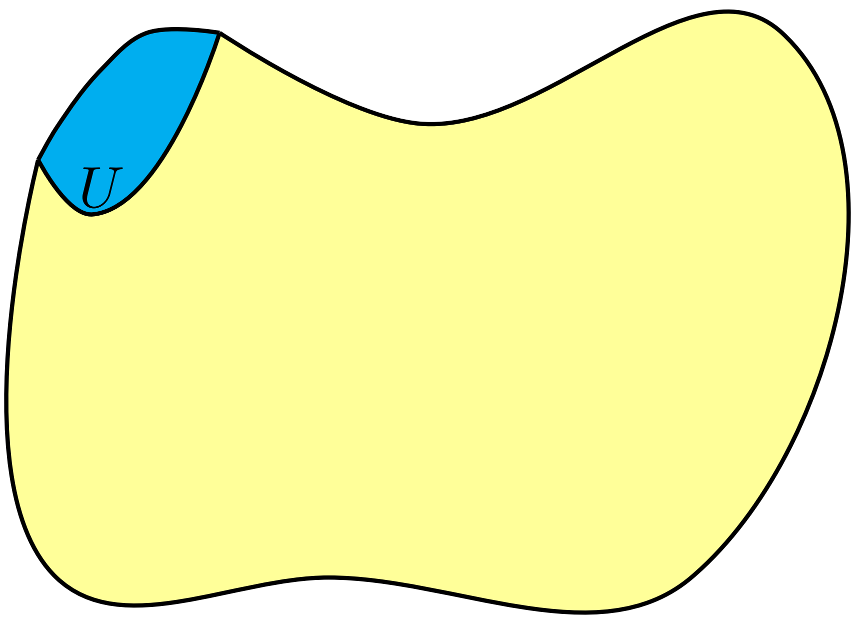
Assume: $h \in H^{1+}(B)$ with $h|_{\partial B} \neq 0$.

$\Longrightarrow\ \exists\, U \subset B$ (open, connected) with $\overline{U} \cap \partial B \neq \emptyset$ and $\operatorname{sgn}(h)|_{\overline{U}} = \text{const} \neq 0$.

Pick $S \subset \overline{U} \cap \partial B$ and an open ball $\Omega_1 \subset U$. Set $\Omega_2 = B \setminus \overline{U}$.



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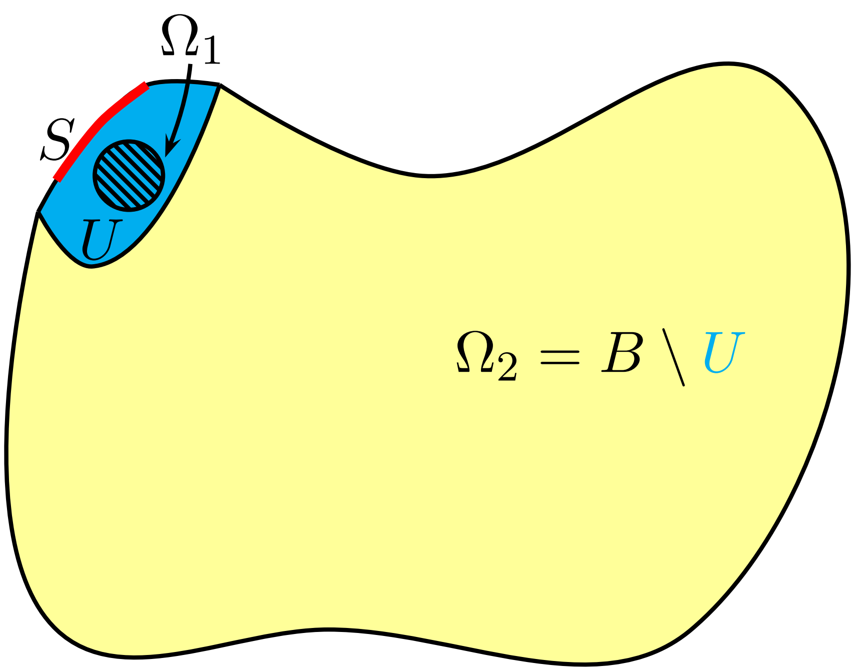

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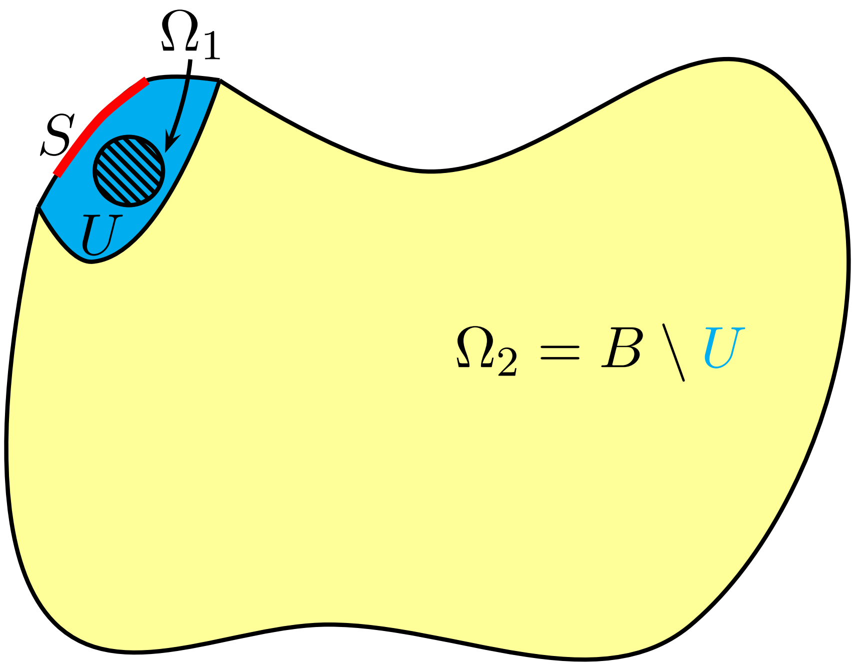

By Gebauer's theorem

$$\Big| \int_B h \, |\nabla u(\sigma, f_n)|^2 \, \mathrm{d}x \Big| \xrightarrow{n\to\infty} \infty.$$

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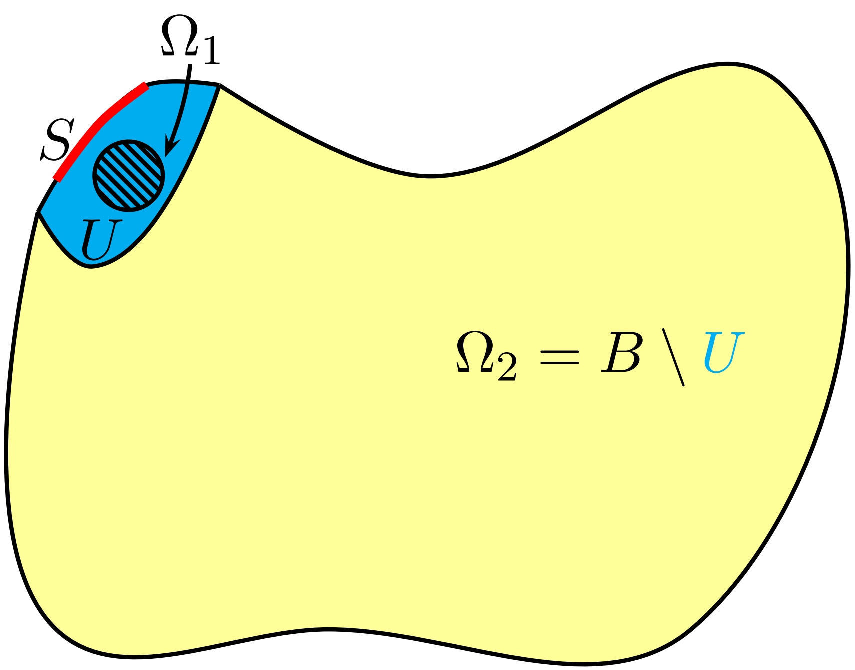

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Assume $h \in \mathsf{N}(F'(\sigma))$. Then,

$$\int_B h |\nabla u(\sigma, f)|^2 \mathrm{d}x = 0 \;\; \forall f \in L^2_\diamond(\partial B)$$

contradicting the above limit.
Hence, $h \notin \mathsf{N}(F'(\sigma))$.

Remainder: $F'(\sigma)[h] = 0 \iff \int_B h \nabla u(\sigma, f) \cdot \nabla u(\sigma, g) \mathrm{d}x = 0 \quad \forall f, g \in L^2_\diamond(\partial B)$

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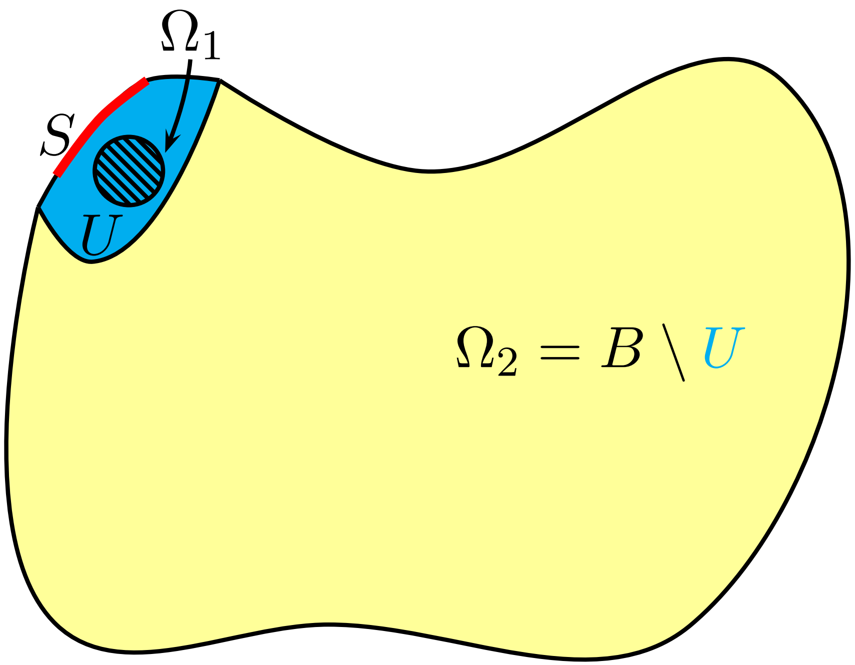

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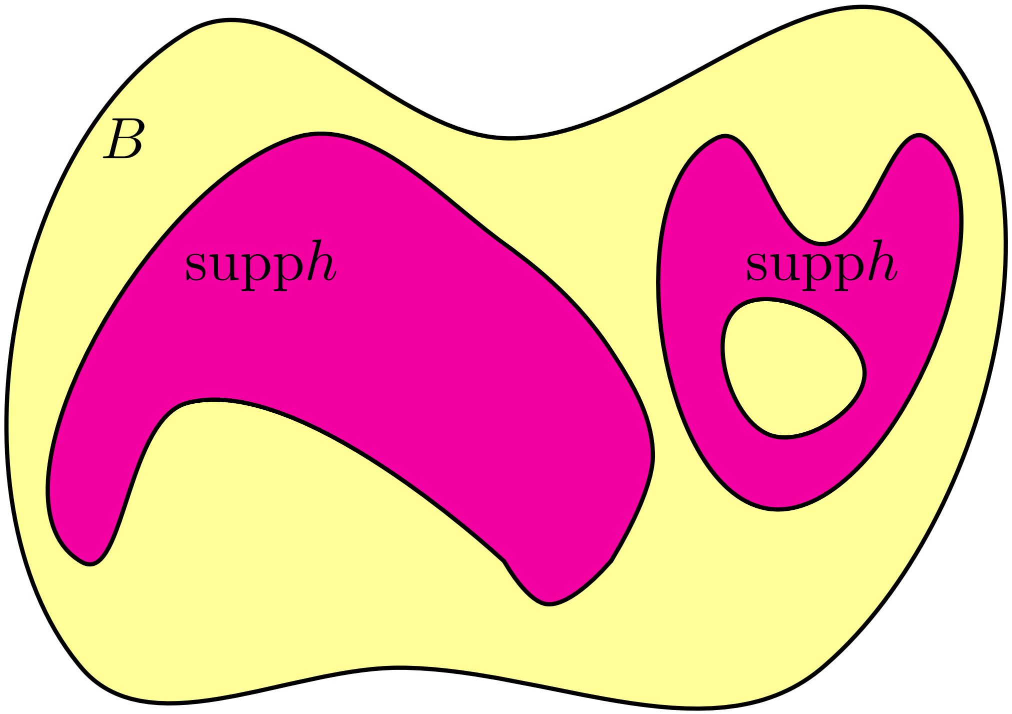
Line of reasoning remains correct if $h|_{\partial B} = 0$ but $\mathrm{sgn}(h)|_{\mathrm{int}(U)} = \mathrm{const} \neq 0$.

On the injectivity of $F'(\sigma)|_{H^{1+}(B)}$ (continued)

$\text{supp}\,h \Subset B$



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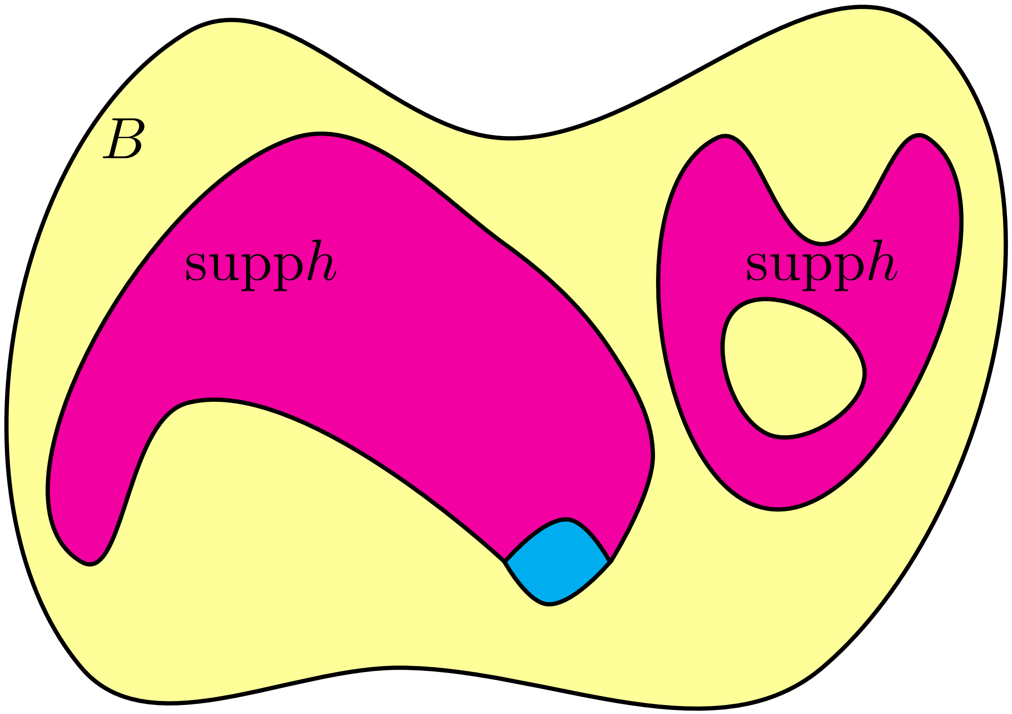

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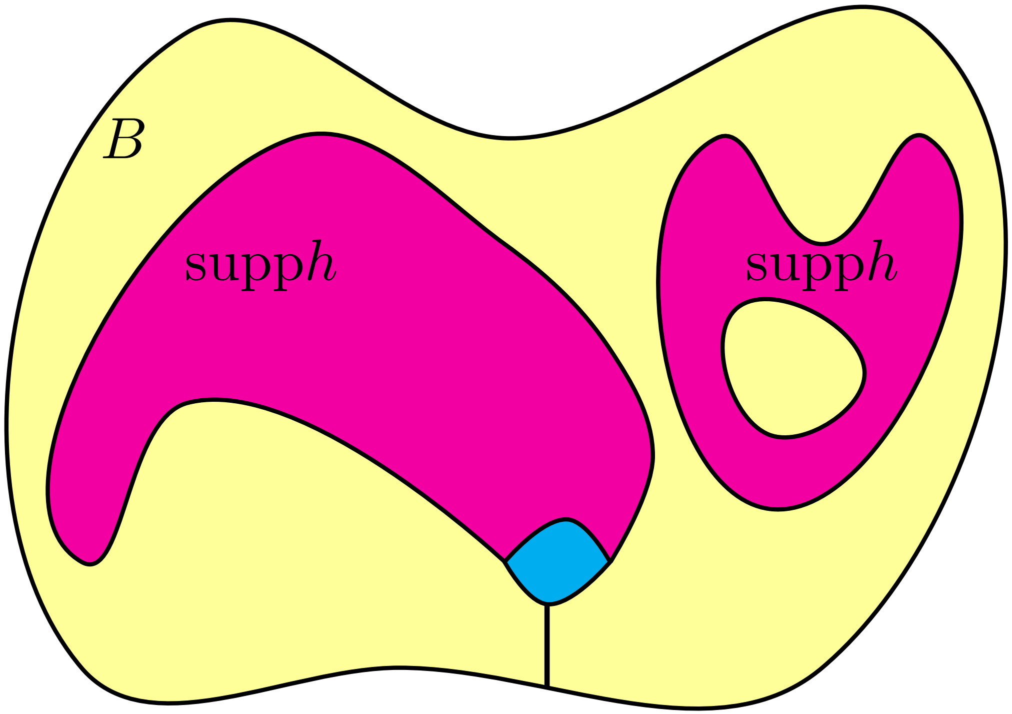

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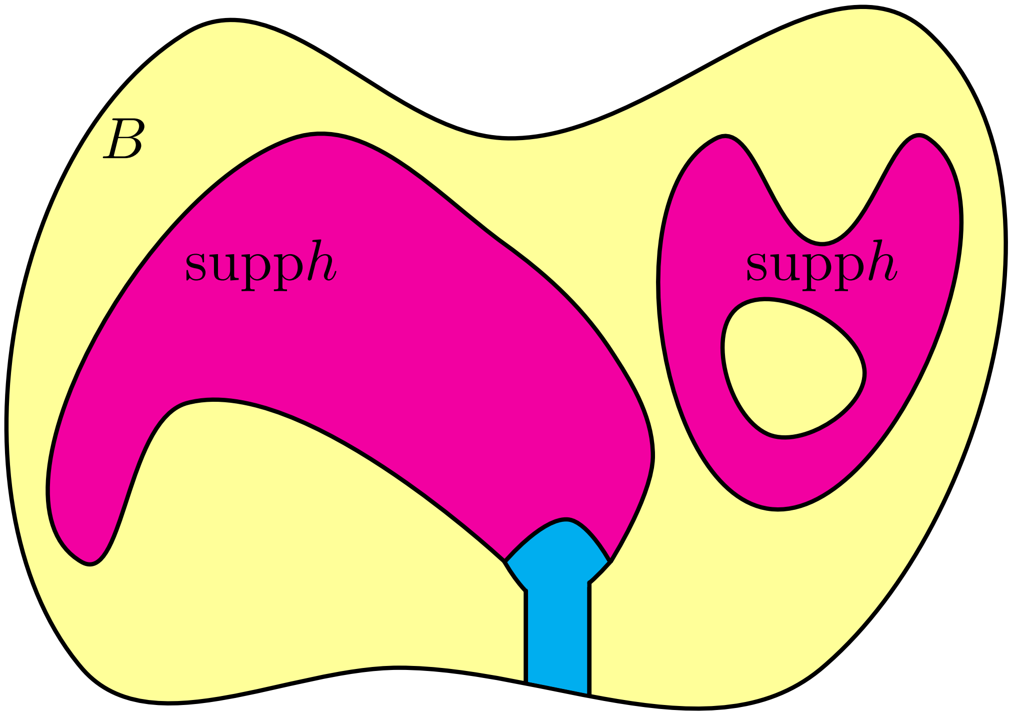

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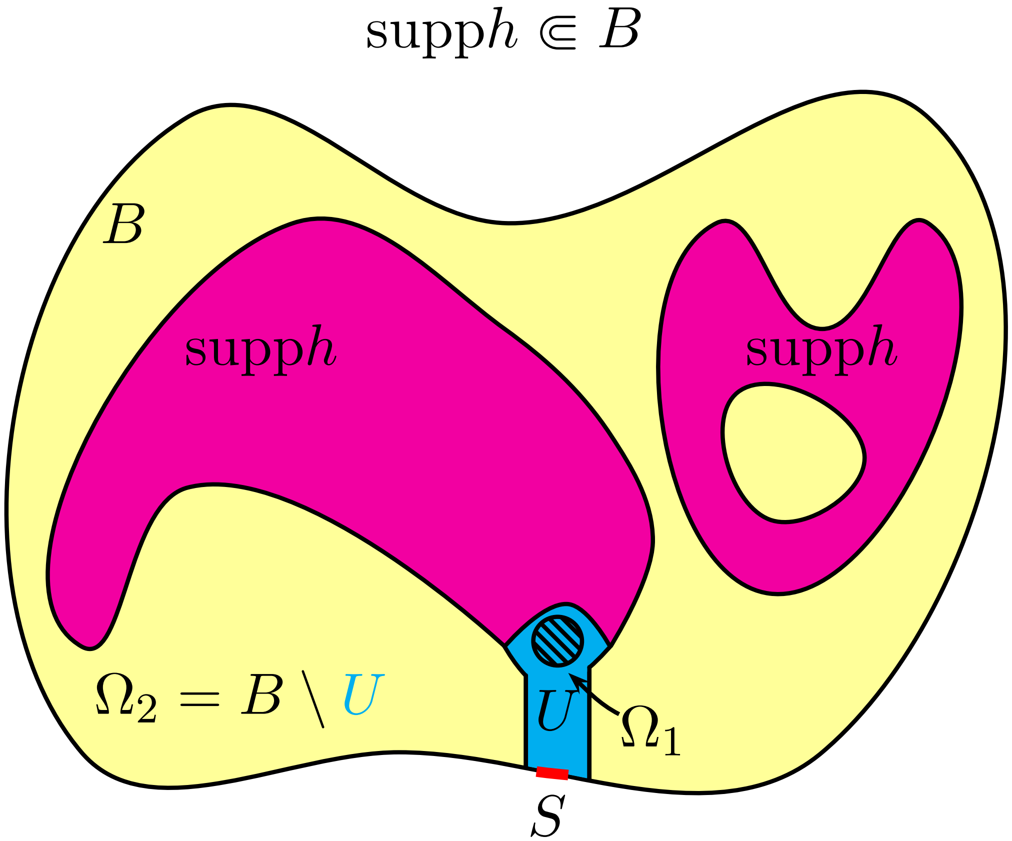

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On the injectivity of $F'(\sigma)|_{H^{1+}(B)}$: summary

$$\mathcal{H}(\sigma) = \{h \in H^{1+}(B) : h|_{\partial B} \neq 0\}$$

$$\bigcup \{h \in H^{1+}(B) : \exists U \subset \operatorname{supp} h \text{ open}, \ \partial U \cap \partial \operatorname{supp} h \neq \emptyset, \ \operatorname{sgn}(h)|_{\operatorname{int}(U)} = \text{const} \neq 0\}$$

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Theorem $\mathcal{H}(\sigma) \cap \mathsf{N}\big(F'(\sigma)\big) = \emptyset$

On the injectivity of $F'(\sigma)|_{H^{1+}(B)}$: conductivities not in $\mathcal{H}(\sigma)$

Unfortunately, there exist pathological h's for which we cannot decide whether $F'(\sigma)[h] \neq 0$ or not.

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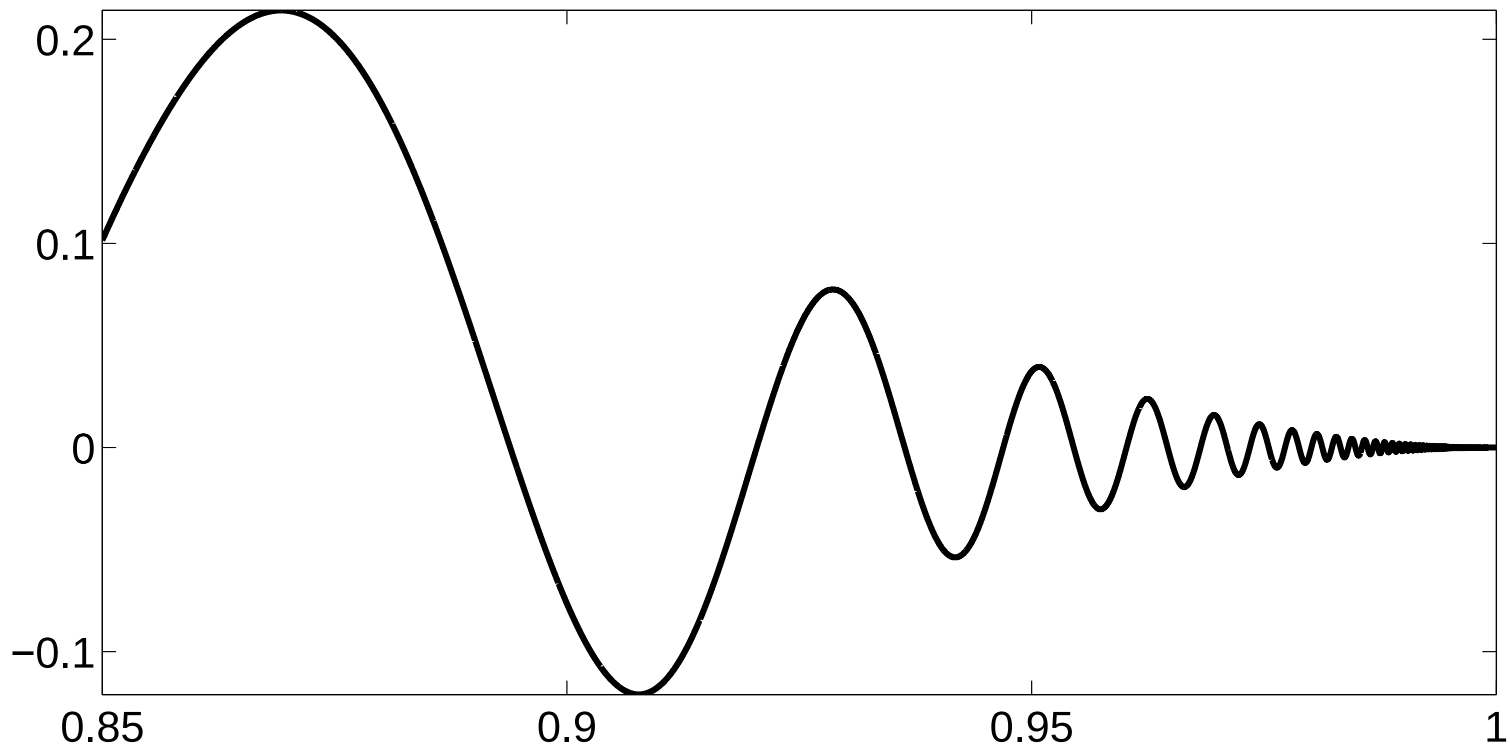
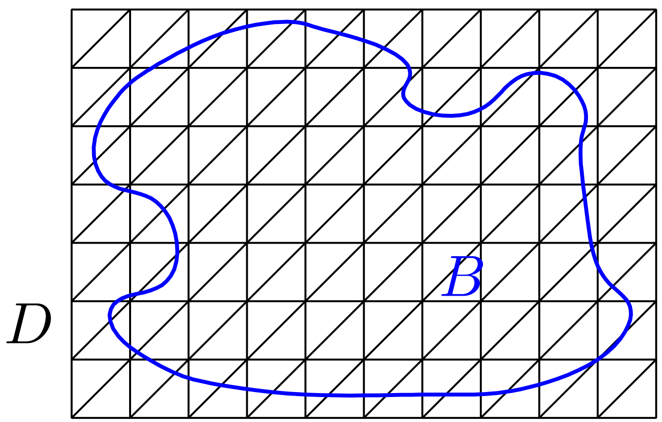
Example Let B the circular disc with radius 1 centered about the origin. Consider $h\colon B \to \mathbb{R}$, $h(x) = r(|x|)$, where r is sketched below:



TCC for finite dimensional spaces

Define a Finite Element space V_ℓ as follows:



$$V_\ell := R_B P_\ell E_D H^{1+}(B) \subset H^{1+}(B)$$

Observe: $V_\ell \cap \mathsf{N}\big(F'(\sigma)\big) = \emptyset$

$$F\colon V_\ell^+ \subset L^\infty(B) \to \mathfrak{L}\big(L^2_\diamond(\partial B)\big)$$

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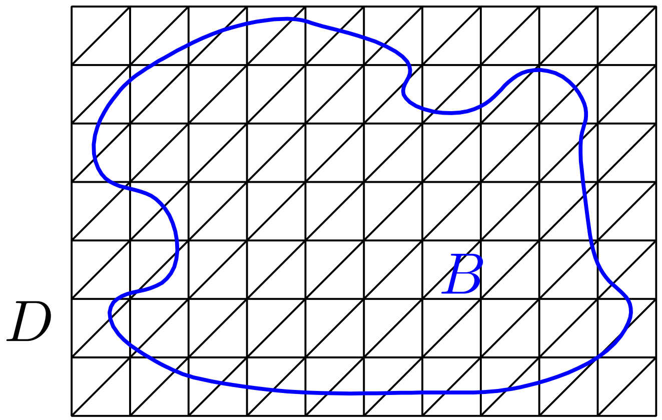

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Theorem If $\sigma_\ell \in \mathrm{int}(V_\ell^+)$ then there is a ball $U(\sigma_\ell) \subset V_\ell$ such that

$$\big\|F(\tau_\ell) - F(\gamma_\ell) - F'(\gamma_\ell)[\tau_\ell - \gamma_\ell]\big\|_{\mathfrak{L}(L^2_\diamond(\partial B))} \leq \mathrm{C}_\ell \, \|\tau_\ell - \gamma_\ell\|_{H^{1+}(B)} \|F(\tau_\ell) - F(\gamma_\ell)\|_{\mathfrak{L}(L^2_\diamond(\partial B))}$$

for any $\tau_\ell, \gamma_\ell \in U(\sigma_\ell)$ where

$$\mathrm{C}_\ell \sim \sup\left\{ \frac{\|F'(\xi_\ell)[h_\ell]\|_{L^2_\diamond(\partial B)\to H^1_\diamond(B)}}{\|F'(\xi_\ell)[h_\ell]\|_{L^2_\diamond(\partial B)\to L^2_\diamond(\partial B)}} : (\xi_\ell, h_\ell) \in U(\sigma_\ell) \times V_\ell \right\}.$$

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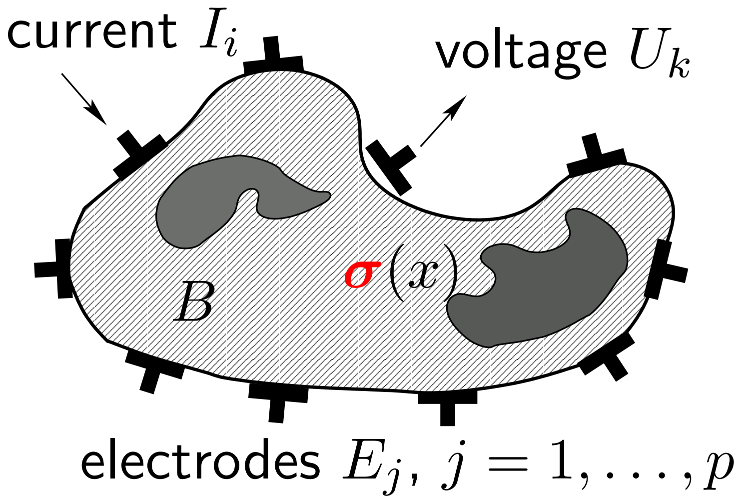

$$\mathrm{div}(\sigma \nabla u) = 0 \quad \text{in } B$$

$$u + z_j\, \sigma \partial_{\mathbf{n}} u = U_j \quad \text{on } E_j$$

$$\sigma \partial_{\mathbf{n}} u = 0 \quad \text{on } \partial B \setminus \cup_j E_j$$

$$\frac{1}{|E_j|} \int_{E_j} \sigma \partial_{\mathbf{n}} u \, \mathrm{d}S = I_j = f|_{E_j}$$

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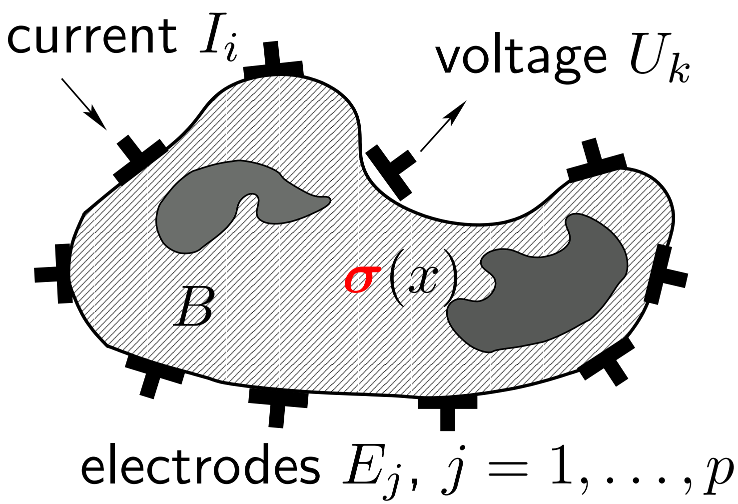

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Given $f \in \mathcal{E}_p := \left\{ \mathbf{j} \in \operatorname{span}\{\chi_{E_1}, \ldots, \chi_{E_p}\} \colon \int_{\partial B} \mathbf{j} \,\mathrm{d}S = 0 \right\} \subset L^2_\diamond(\partial B)$ find $(u, U) \in H^1(B) \oplus \mathcal{E}_p$:

$$b_\sigma\big((u, U), (w, W)\big) = \int_{\partial B} f W \,\mathrm{d}S \quad \forall (w, W) \in H^1(B) \oplus \mathcal{E}_p \tag{1}$$

where

$$b_\sigma\big((v, V), (w, W)\big) = \int_B \sigma \nabla v \cdot \nabla w \,\mathrm{d}x + \sum_{j=1}^{p} \frac{1}{z_j} \int_{E_j} (v - V)(w - W) \,\mathrm{d}S$$

(Existence & Uniqueness: Cheney, Isaacson & Somersalo, 1992)

The forward operator

$$F_p \colon D(F) \subset L^\infty(B) \to \mathfrak{L}\big(\mathcal{E}_p\big), \quad \sigma \mapsto \big\{f \mapsto U\big\}, \quad F_p(\sigma)f = U,$$

where U is the second component of the soution of (1).

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Conjecture If $V \subset L^\infty(B)$ with $\dim V \leq p(p-1)/2$ then

$$F_p : D(F) \cap V \subset V \to \mathfrak{L}\big(\mathcal{E}_p\big) \quad \text{is injective.}$$

TCC for CEM

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is a mapping between finite dimensional spaces TCC holds locally about $\sigma \in \operatorname{int}(D(F) \cap V)$ whenever $F_p'(\sigma) \in \mathfrak{L}\big(V, \mathfrak{L}\big(\mathcal{E}_p\big)\big)$ is injective.

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$$F_p'(\sigma)[\eta] = 0 \quad \Longleftrightarrow \quad \int_B \eta \nabla u(\sigma, f) \cdot \nabla u(\sigma, g) \mathrm{d}x = 0 \quad \forall f,\, g \in \mathcal{E}_p$$

where $u = u(\sigma, f) \in H^1(B)$ is the first component of the solution of

$$b_\sigma\big((u, U), (w, W)\big) = \int_{\partial B} f W \mathrm{d}S \quad \forall (w, W) \in H^1(B) \oplus \mathcal{E}_p.$$

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Conjecture If $V \subset L^\infty(B)$ with $\dim V \le p(p-1)/2$ then $F_p'(\sigma)$ is injective for any $\sigma \in \operatorname{int}(D(F) \cap V)$.

The Tangential Cone Condition (TCC)
EIT: Continuous Model
EIT: Complete Electrode Model
▷ Conclusion

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What to remember from this talk

- TCC is a vital ingredient for the convergence analysis of iterative regularization schemes for nonlinear ill-posed problems.
- As the forward operator of the continuous model for EIT is injective, a necessary prerequisite for the TCC to hold is the injectivity of the Frechét derivative. We have shown that only 'pathological' elements can possibly be in the Null space. If we restrict the conductivities to a finite dimensional space, say, a finite element space, then TCC holds. Unfortunately, this is not an adequate setting for the continuous model.
- On the other hand, the CEM offers only finitely many independent measurements. Therefore, a finite dimensional setting is necessary to have injectivity of the forward operator. We conjectured that injectivity of the forward operator and its derivative hold if the number of DOF of the searched-for conductivity is at most the number of independent measurements. If the conjectures apply then TCC holds.

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Thank you for your attention!