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A new view on phantom views

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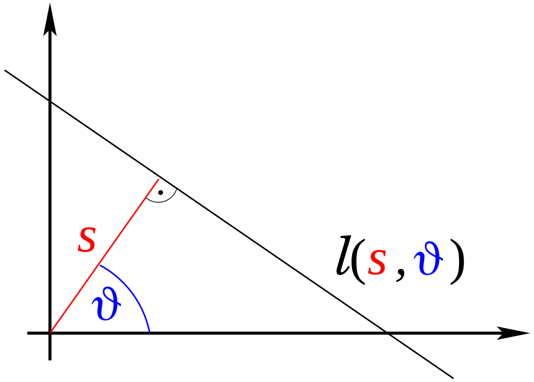
Introduction: FBA augmented by phantom views

2D-Radon transform (parallel scanning geometry)

$$\mathbf{R}f(s,\vartheta) \; := \int_{l(s,\vartheta)\cap\Omega} f(x)\,\mathrm{d}\sigma(x)$$



tomographic inversion: $\mathbf{R}f(s,\vartheta) = g(s,\vartheta)$

$$\mathbf{R}\colon L^2(\Omega) \to L^2(Z), \quad Z = [-1,1] \times [0,2\pi]$$

Inversion formula

$$f = \frac{1}{4\pi}\mathbf{R}^*(\Lambda \otimes I)\mathbf{R}f$$

$\mathbf{R}^*\colon L^2(Z) \to L^2(\Omega)$ Backprojection operator

$$\mathbf{R}^*g(x) = \int_0^{2\pi} g(x^t\omega(\vartheta), \vartheta)\,\mathrm{d}\vartheta, \quad \omega(\vartheta) = (\cos\vartheta, \sin\vartheta)^t$$

$\Lambda\colon H^\alpha(\mathbb{R}) \to H^{\alpha-1}(\mathbb{R})$ Riesz potential

$$\widehat{\Lambda u}(\xi) = |\xi|\widehat{u}(\xi).$$

Filtered backprojection algorithm (FBA)

discrete Radon data $D = \{\mathbf{R}f(kh, jh_\vartheta) : k = -q, \dots, q,\ j = 0, \dots, 2p-1\}$,

$h = 1/q, \quad h_\vartheta = \pi/p$

$$f_{\mathsf{FBA}}(x) := \frac{1}{4\pi} \mathbf{R}^*_{h_\vartheta} (\mathrm{I}_h \Lambda E_h \otimes I) \mathbf{R} f(x)$$

where

E_h, I_h generalized interpolation operators

and

$$\mathbf{R}^*_{h_\vartheta} g(x) := h_\vartheta \sum_{j=0}^{2p-1} g(x^t \omega(\vartheta_j), \vartheta_j), \quad \vartheta_j = jh_\vartheta$$

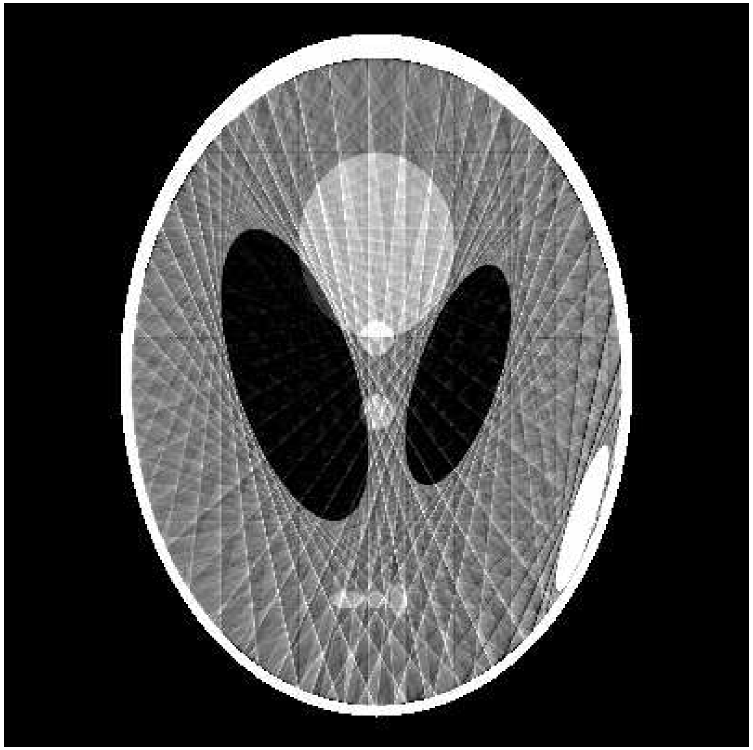
Remark: The action of $\mathrm{I}_h \Lambda E_h$ can be implemented as a convolution (filtering) followed by an interpolation. The convolution kernel (reconstruction filter) depends on I_h and E_h.

Angular under-sampling causes artifacts

FBA works well for standard parallel scanning geometry under optimal sampling, that is,

$$h \approx h_\vartheta \quad (p \approx \pi q).$$

In case of severe angular under-sampling ($h_\vartheta \gg h$) the FBA reconstructions are corrupted by heavy streak artifacts:



$h = h_\vartheta/50$

Introducing phantom views

Lewitt et al.(1978) suggested to increase the angular sampling by interpolating the Radon data linearly w.r.t. the angular variable:

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$$f_{\mathsf{PhanFBA}(R)}(x) := \frac{1}{4\pi}\mathbf{R}^*_{h_\vartheta/R}(\mathrm{I}_h \Lambda E_h \otimes T_{h_\vartheta})\mathbf{R}f(x), \quad R \in \mathbb{N}$$

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$$f_{\mathsf{PhanFBA}(R)}(x) := \underbrace{\frac{1}{4\pi}\mathbf{R}^*_{h_\vartheta/R}(\mathrm{I}_h \Lambda E_h \otimes I)}_{\text{Step 2}}\underbrace{(I \otimes T_{h_\vartheta})}_{\text{Step 1}}\mathbf{R}f(x), \quad R \in \mathbb{N}$$

Step 1: linear interpolation of Radon data in angular variable
Step 2: standard FBA with angular steps size h_ϑ/R.

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FBA PhanFBA(2) PhanFBA(5)

Phantom views reduce streak artifacts

A different view on PhanFBA I

With $\Phi := (\mathrm{I}_h \Lambda E_h \otimes T_{h_\vartheta})\mathbf{R}f$ and $\vartheta_{j+\frac{\ell}{R}} := (j + \frac{\ell}{R})h_\vartheta$ we have that

$$f_{\mathsf{PhanFBA}(R)}(x) = \frac{h_\vartheta}{R} \sum_{j=0}^{2p-1} \sum_{\ell=0}^{R-1} \Phi(x^t \omega(\vartheta_{j+\ell/R}), \vartheta_{j+\ell/R})$$

$$= \frac{h_\vartheta}{R} \sum_{j=0}^{2p-1} \sum_{\ell=0}^{R-1} \left(\left(1 - \frac{\ell}{R}\right) \Phi(x^t \omega(\vartheta_{j+\ell/R}), \vartheta_j) + \frac{\ell}{R} \Phi(x^t \omega(\vartheta_{j+\ell/R}), \vartheta_{j+1}) \right)$$

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$$= \frac{1}{R} \left(f_{\mathsf{FBA}}(x) + \sum_{\ell=1}^{R-1} \left(1 - \frac{\ell}{R}\right) \left(f_{\mathsf{FBA}}(U_{\frac{\ell}{R} h_\vartheta} x) + f_{\mathsf{FBA}}(U_{-\frac{\ell}{R} h_\vartheta} x) \right) \right)$$

where $U_\varphi \in \mathbb{R}^{2\times 2}$ is rotation by angle φ.

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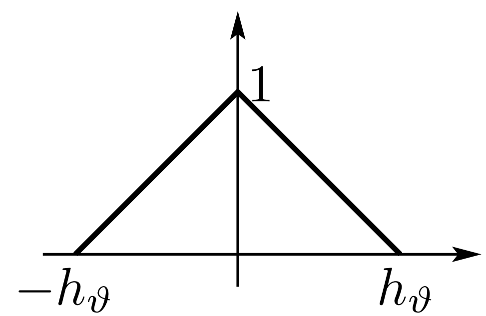
$$= \frac{1}{R}\left(f_{\mathsf{FBA}}(x) + \sum_{\ell=1}^{R-1}\left(1-\frac{\ell}{R}\right)\left(f_{\mathsf{FBA}}(U_{\frac{\ell}{R}h_\vartheta}x) + f_{\mathsf{FBA}}(U_{-\frac{\ell}{R}h_\vartheta}x)\right)\right)$$

where $U_\varphi \in \mathbb{R}^{2\times 2}$ is rotation by angle φ. Hence,

$$f_{\mathsf{PhanFBA}(R)}(0) = f_{\mathsf{FBA}}(0)$$

and $f_{\mathsf{PhanFBA}(R)}(x)$ is the trapezoidal sum with step size $\frac{1}{R}$ applied to

$$\frac{1}{h_\vartheta}\int_{-h_\vartheta}^{h_\vartheta} f_{\mathsf{FBA}}(U_\varphi x)\, B_{h_\vartheta}(\varphi)\,\mathrm{d}\varphi$$



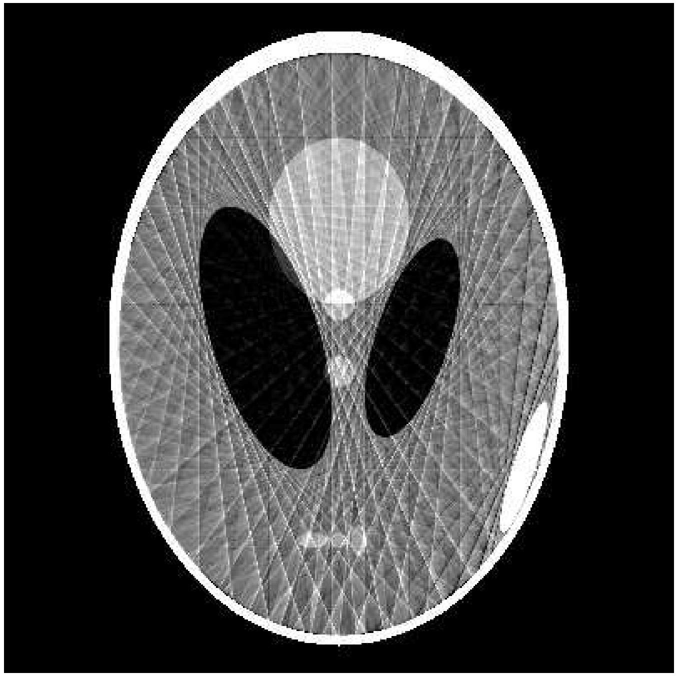
where B_{h_ϑ} is the linear B-Spline w.r.t. $[-h_\vartheta, h_\vartheta]$.

A different view on PhanFBA II

As PhanFBA(R) is an angular average of FBA,

$$f_{\mathsf{PhanFBA}(R)}(x) \approx \frac{1}{h_\vartheta} \int_{-h_\vartheta}^{h_\vartheta} f_{\mathsf{FBA}}(U_\varphi x)\, B_{h_\vartheta}(\varphi)\, \mathrm{d}\varphi,$$

those edges being tangent to a circle centered about the origin are not blurred. The more transversally an edge intersects such a circle the more it gets blurred.



FBA

PhanFBA(5)

Phantom views increase angular convergence rate

As $R \to \infty$,

$$f_{\mathsf{PhanFBA}(R)}(x) = \frac{1}{4\pi}\mathbf{R}^*_{h_\vartheta/R}(\mathrm{I}_h\Lambda E_h \otimes T_{h_\vartheta})\mathbf{R}f(x)$$

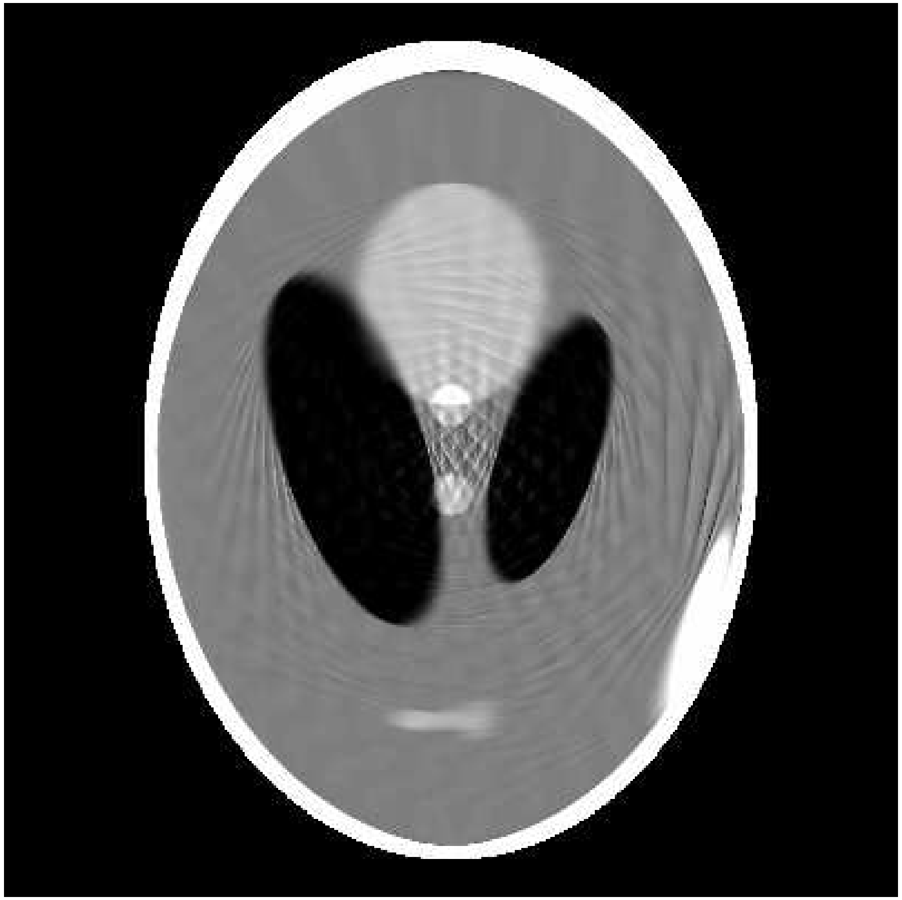
converges to

$$\begin{aligned} f_{\mathsf{PhanFBA}(\infty)}(x) &:= \frac{1}{4\pi}\mathbf{R}^*(\mathrm{I}_h\Lambda E_h \otimes T_{h_\vartheta})\mathbf{R}f(x) \\ &= \frac{1}{h_\vartheta}\int_{-h_\vartheta}^{h_\vartheta} f_{\mathsf{FBA}}(U_\varphi x)\, B_{h_\vartheta}(\varphi)\,\mathrm{d}\varphi. \end{aligned}$$

Remark: The evaluation of $f_{\mathsf{PhanFBA}(\infty)}(x)$ can be organized as standard FBA with an additional multiplication of the filtered data by a sparse matrix.

PhanFBA(R) vs. PhanFBA(∞)

PhanFBA(5)



PhanFBA(∞)

PhanFBA(∞) vs. FBA: Convergence rates

Let $f \in H_0^\alpha(\Omega)$. Then,

$$\left\| \frac{1}{4\pi} \mathbf{R}^*_{h_\vartheta} (\mathrm{I}_h \Lambda E_h \otimes I) \mathbf{R} f - f \right\|_{L^2} \lesssim \left(h^{\min\{\alpha_{\max}, \alpha\}} + h_\vartheta^\alpha + h_\vartheta h^{\min\{\alpha_{\max}, \alpha-1\}} \right) \|f\|_\alpha, \quad \alpha \geq 1$$

$$\left\| \frac{1}{4\pi} \mathbf{R}^* (\mathrm{I}_h \Lambda E_h \otimes T_{h_\vartheta}) \mathbf{R} f - f \right\|_{L^2} \lesssim \left(h^{\min\{\alpha_{\max}, \alpha\}} + h_\vartheta^{\min\{5/2, \alpha\}} \right) \|f\|_\alpha, \quad \alpha > 1/2$$

$$\alpha_{\max} = \begin{cases} 3/2 : & \text{Shepp-Logan with piecewise constant interpolation} \\ 2 : & \text{Shepp-Logan with piecewise linear interpolation} \\ 5/2 : & \text{mod. Shepp-Logan with piecewise linear interpolation} \end{cases}$$

PhanFBA(∞) vs. FBA: Convergence rates

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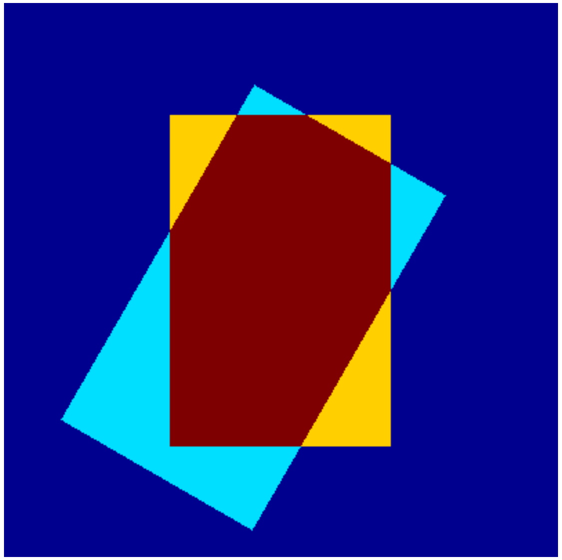
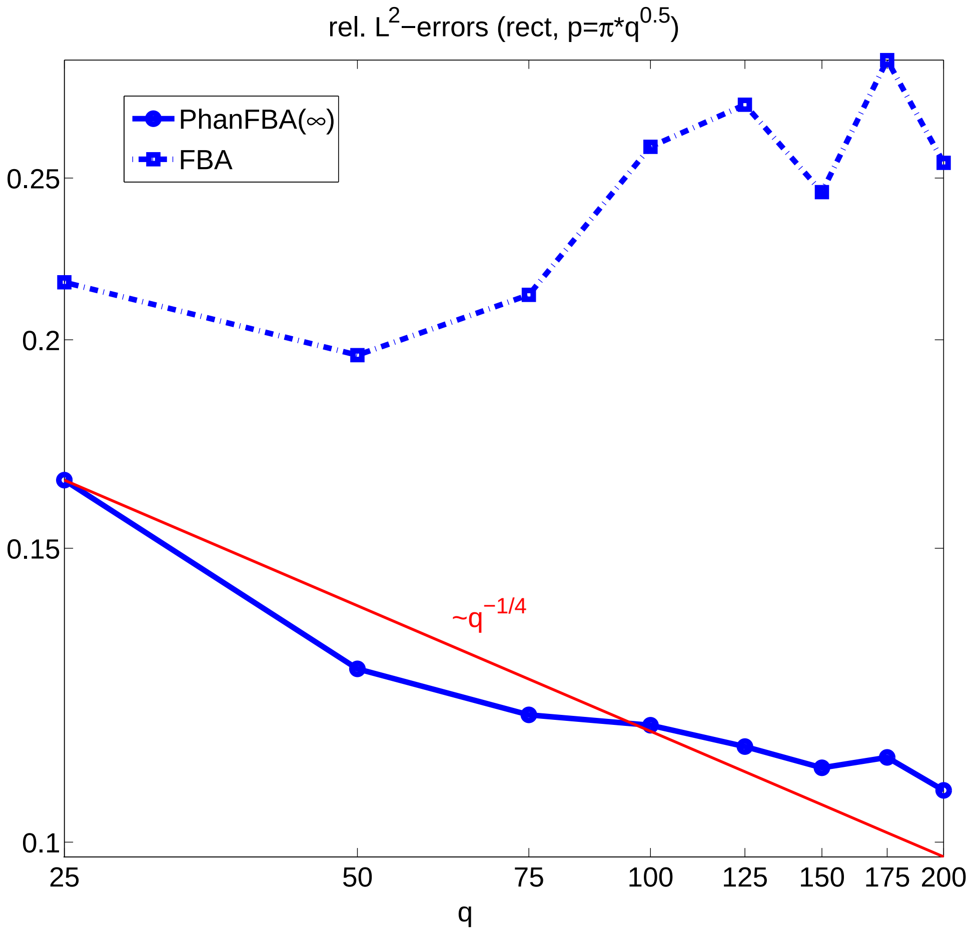
$f \in H_0^\alpha(\Omega)$, $\alpha \approx \frac{1}{2}$, and $h_\vartheta \approx \sqrt{h}$

$\Longrightarrow$ $\mathsf{err}_{\mathsf{FBA}} \approx$ const., $\mathsf{err}_{\mathsf{PhanFBA}} \approx h^{1/4}$

PhanFBA(∞) vs. FBA: A numerical comparison



in $H_0^\alpha(\Omega)$, $\alpha < 1/2$

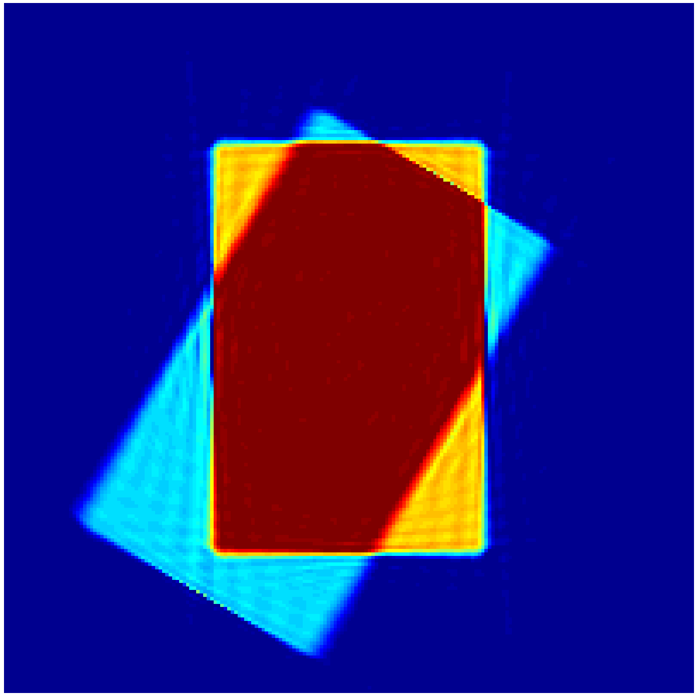


$$h = \frac{1}{q}, \quad h_\vartheta = \frac{\pi}{p} \text{ where } p = \lceil \pi\sqrt{q} \rceil$$

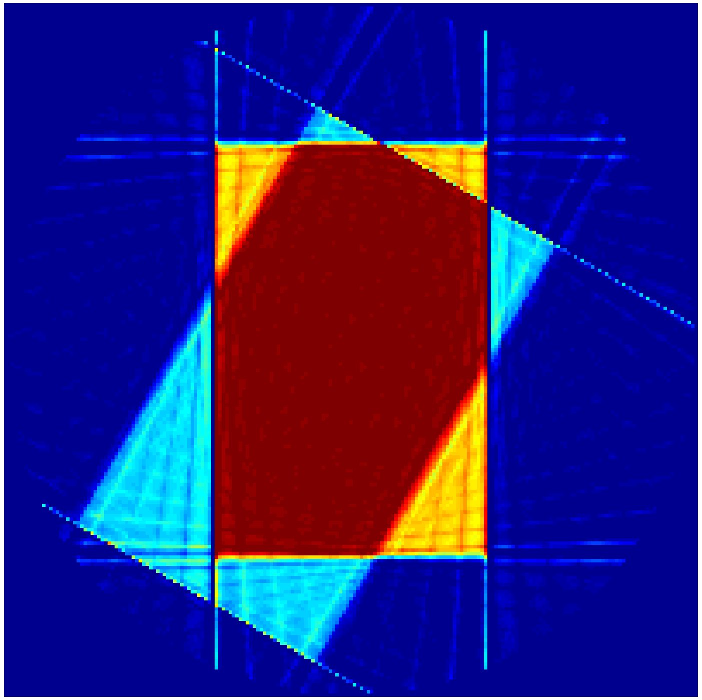
$$e(q) := \Big(\sum_{x \in \mathcal{X}_q} \big(f_{\text{recon}}(x) - f(x)\big)^2 \Big/ \sum_{x \in \mathcal{X}_q} f(x)^2 \Big)^{1/2}, \quad \mathcal{X}_q = \Omega \cap \mathbb{Z}^2/(800)$$

PhanFBA(∞) vs. FBA: Reconstructions

PhanFBA(∞), q=200, p=45, lin.interpol



FBA, q=200, p=45, lin. interpol



Bibliographical notes

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- R. M. Lewitt, R. H. T. Bates, T. M. Peters
  Image reconstruction from projections II*: Modified backprojection methods*, Optik 50 (1978), 85–109.
- R. R. Galigekere, K. Wiesent, D. W. Holdsworth
  Techniques to alleviate the effects of view aliasing artifacts in computed tomography, Med. Phys. 26 (1999), 896–904.
- A. Rieder, A. Faridani
  The semi-discrete filtered backprojection algorithm is optimal for tomographic inversion, SIAM J. Numer. Anal. 41 (2003), 869–892.
- A. Rieder, A. Schneck
  Optimality of the fully discrete filtered backprojection algorithm for tomographic inversion, Numer. Math. 108 (2007), 151–175.

Conclusion

What to remember from this talk

- PhanFBA(∞) is an angular average of standard FBA.
- Streaks intersecting transversally a circle centered about the origin are diminished. Streaks being tangent to such a circle and a whole neighborhood of the origin remain unaffected by PhanFBA.
- Phantom views increase the angular convergence rate.

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Thank you for your attention!

GAMM 2010

MARCH, 22-26

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of the International Association of Applied Mathematics and Mechanics

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