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Semi-discrete equations in Banach spaces: The approximate inverse approach

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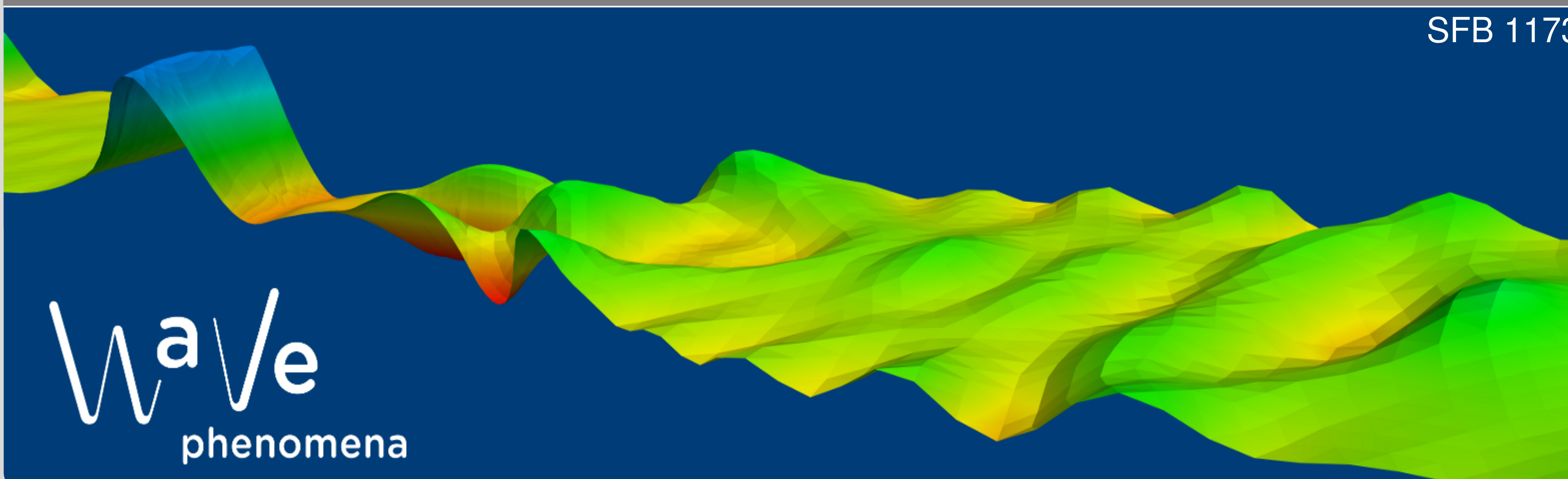


AI in Banach spaces: pros and cons

Pros:

- No geometric assumptions on Banach spaces.
- Discretization effects included into the analysis.

Cons:

- Some components may not be easily available in concrete applications.
- Abstract concept; not easily presented.

Setting the stage

- X, Y real Banach spaces
- $A \in \mathcal{L}(X, Y)$, $\mathsf{R}(A) \neq \overline{\mathsf{R}(A)}$

inverse problem: $Af = g$ where $g \in \mathsf{R}(A)$

Remarks:

- The inverse problem is ill-posed in the sense of Nashed.
- There is no bounded inner inverse for A:
$$\nexists\, B \in \mathcal{L}(Y, X): \; ABA = A.$$
Inner inverses are equation solvers: $\widetilde{f} = Bg$ solves the inverse problem.
- In the Hilbert space setting A^+ (Moore-Penrose inverse) is an inner inverse which is bounded iff $\mathsf{R}(A) = \overline{\mathsf{R}(A)}$.

Modelling including Measurement Process

$\mathbf{\Psi}_n : Y \to \mathbb{R}^n$ observation operator (linear, bounded)

Example: $\psi_i \in Y'$ sensitivity profile of i-th detector

$$\mathbf{\Psi}_n g := \Big(\langle \psi_1, g\rangle_{Y'\times Y}, \ldots, \langle \psi_n, g\rangle_{Y'\times Y}\Big)^\top$$

Semi-discrete problem (SDP): $A_n f_n = g_n$ (No discretization!)

where

$$A_n = \mathbf{\Psi}_n A \in \mathcal{L}(X, \mathbb{R}^n), \quad g_n = \mathbf{\Psi}_n g, \quad f_n \in X$$

- SDP is highly under-determined!
- If X is uniformly convex then the minimum norm solution of SDP is well defined and unique.

The key concept: computing moments

- Instead of trying to find a solution $f_n \in X$ of SDP we aim at computing moments:

$$\langle e_{d,i}, f_n \rangle_{X' \times X} \quad \text{for mollifiers } e_{d,i} \in X', \; i = 1, \ldots, d.$$

- Then, we recover an approximate solution $E_d f_n$ of SDP by

$$E_d f_n := \sum_{i=1}^{d} \langle e_{d,i}, f_n \rangle_{X' \times X} \, b_{d,i}$$

  where $\{b_{d,i}\}_{i=1}^d \subset X$.

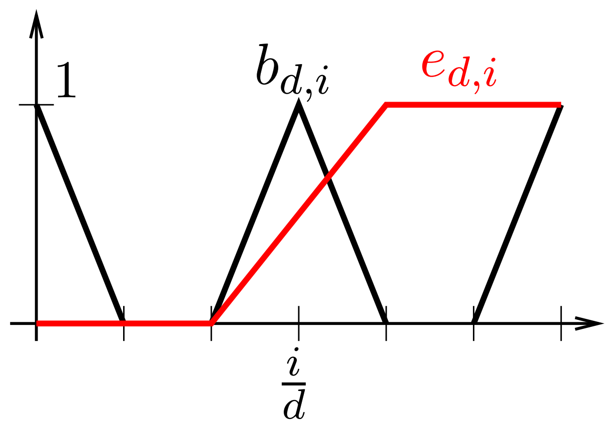
- $\{e_{d,i}\}_{i=1}^d \subset X'$ and $\{b_{d,i}\}_{i=1}^d \subset X$ have to be intertwined such that E_d satisfies the *mollifier property*

$$\lim_{d \to \infty} \|E_d w - w\|_X = 0 \quad \text{for any } w \in X.$$

Mollifier property: an example

- $X = (\mathcal{C}([0,1]), \|\cdot\|_\infty)$, $X' \simeq \mathrm{BV}_*(0,1)$: $f \mapsto \langle \mu, f \rangle_{X' \times X} := \int_0^1 f(x) \mathrm{d}\mu(x)$.
- Define

$$E_d f = \sum_{i=0}^{d} \langle e_{d,i}, f \rangle_{X' \times X} \, b_{d,i} \quad \text{where} \quad \langle e_{d,i}, f \rangle_{X' \times X} = \frac{2}{d} \int_{(i-1)/d}^{(i+1)/d} f(x) \mathrm{d}x.$$



Then,

$$\lim_{d \to \infty} \|f - E_d f\|_\infty = 0 \quad \text{for any } f \in \mathcal{C}(0,1).$$

Moreover, $\|f - E_d f\|_\infty \le C_E \, d^{-\alpha} \|f\|_{\mathcal{C}^\alpha(0,1)}$, $0 < \alpha \le 1$.

Reconstruction kernels: the basic idea

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To evaluate $E_d f_n$ we need to calculate the moments $\langle e_{d,i}, f_n \rangle_{X' \times X}$.

Let the equations

$$A_n^* v_{d,i}^n = e_{d,i}, \quad i = 1, \dots, d, \tag{1}$$

have solutions. Then,

$$\langle e_{d,i}, f_n \rangle_{X' \times X} = \langle v_{d,i}^n, A_n f_n \rangle_2 = \langle v_{d,i}^n, g_n \rangle_2$$

and defining

$$\widetilde{A}_{n,d} \colon \mathbb{R}^n \to X \,, \quad \widetilde{A}_{n,d} \alpha := \sum_{i=1}^{d} \langle v_{d,i}^n, \alpha \rangle_2 \, b_{d,i}$$

we obtain

$$\widetilde{A}_{n,d} \, A_n = E_d.$$

- $\widetilde{A}_{n,d}$ *approximate inverse* of A_n, $\quad v_{d,i}^n$ *reconstruction kernel*

However, (1) cannot be expected to have solutions.

(Louis, Maass 1990, Louis 1996)

Surrogate kernels from the underlying continuous setting

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Let A be injective. Given $\varepsilon > 0$ and $e \in X'$.

- If X is reflexive, $\mathcal{R}(A^*)$ is dense in X' and we find $v \in Y'$:

$$\|A^* v - e\|_{X'} \leq \varepsilon.$$

- If X is arbitrary, $\mathcal{R}(A^*)$ is weak-★ dense in X'.
  Let $f \in X$ denote the (unique) solution of $Af = g$.
  Then, we find $v = v(f) \in Y'$:

$$|\langle A^* v - e, f\rangle_{X' \times X}| \leq \varepsilon \, \|f\|_X.$$

We call v a continuous kernel with inaccuracy ε (related to A, f, and e).

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Continuous kernels: an example

- $X = (\mathcal{C}([0,1]), \|\cdot\|_\infty)$, $X' \simeq \mathrm{BV}_*(0,1)$
- $A\colon X \to X$, $Af(x) = \int_0^x f(t)\mathrm{d}t$
- $A^*\colon X' \to X'$, $A^*\mu(\xi) = \mu(1)\xi - A\mu(\xi)$.
- Let $\eta \in X'$ with $\int_0^1 \eta(t)\mathrm{d}t = 1$ and $\eta(1) = 0$. Define $e(x) := A\eta(x)$.
  Then, $v(x) = -\eta(x)$ satisfies
  $$A^*v(\xi) = v(1)\xi - Av(\xi) = -\eta(1)\xi + A\eta(\xi) = e(\xi).$$
- Let $\eta_\gamma = \chi_{[z-\gamma, z+\gamma[}/(2\gamma)$ for a fixed $z \in]0,1[$ with $\gamma > 0$ suff. small. Then,
  $$\langle e_\gamma, f\rangle_{X'\times X} = \langle v_\gamma, Af\rangle_{X'\times X} = -\int_0^1 Af(t)\mathrm{d}\eta_\gamma(t)$$
  $$= \frac{Af(z+\gamma) - Af(z-\gamma)}{2\gamma} \xrightarrow{\gamma\to 0} (Af)'(z) = f(z).$$

Surrogate kernels from the underlying continuous setting (cont'd)

Having continuous kernels $v_{d,i}$ satisfying

$$|\langle A^* v_{d,i} - e_{d,i}, f\rangle_{X' \times X}| \leq \varepsilon_{d,i} \, \|f\|_X, \quad i = 1, \dots, d,$$

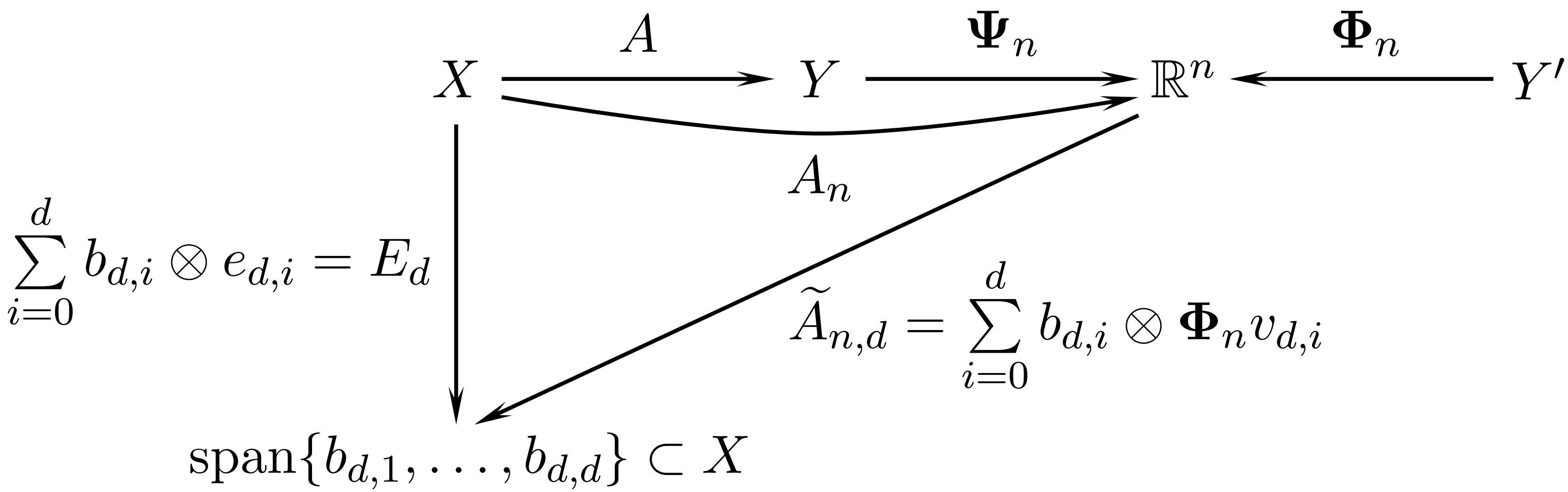
at our disposal we define the approximate inverse with surrogate kernels

$$\widetilde{A}_{n,d}\alpha := \sum_{i=1}^{d} \langle \boldsymbol{\Phi}_n v_{d,i}, \alpha\rangle_2 \, b_{d,i}.$$

Here, $\boldsymbol{\Phi}_n : Y' \to \mathbb{R}^n$ is linear and continuous and needs to be defined such that

$$\lim_{\substack{n\to\infty \\ d\to\infty}} \|\widetilde{A}_{n,d} A_n f - f\|_X = 0 \quad \text{for any } f \in X.$$

Interlude: overview of the involved operators



$$E_d \approx \widetilde{A}_{n,d} A_n$$

Convergence

Theorem: If all building blocks are chosen adequately then

$$\|\widetilde{A}_{n,d}A_n f - f\|_X \lesssim \|f - E_d f\|_X + \sigma(d)\big(\rho_n \max_{1\le i\le d}\|v_{d,i}\|_{Y'} + \max_{1\le i\le d}\varepsilon_{d,i}\big)$$

where $\rho_n \to 0$ as $n \to \infty$.

Four error components:

1. Discretization error in X by mollification operator,
2. Discretization error in Y by observation operator and
3. Error due to the surrogate kernels: $\rho_n = \rho_n(Af, \mathbf{\Psi}_n, \mathbf{\Phi}_n)$,
4. Accuracy of the continuous kernels ($A^* v_{d,i} \approx e_{d,i}$).

$$\Big\|\sum_{i=1}^{d}\alpha_i\, b_{d,i}\Big\|_X \le \sigma(d)\max_{1\le i\le d}|\alpha_i|, \quad \alpha \in \mathbb{R}^d$$

Convergence (cont'd)

Corollary: Choosing $d = d(n) = d(n, f)$ such that $d(n) \to \infty$ as $n \to \infty$ as well as

$$\sigma\big(d(n)\big)\, \rho_n \max_{1 \le i \le d(n)} \|v_{d(n),i}\|_{Y'} \to 0 \qquad \text{as } n \to \infty$$

and

$$\sigma\big(d(n)\big) \max_{1 \le i \le d(n)} \varepsilon_{d(n),i} \to 0 \qquad \text{as } n \to \infty$$

we have the convergence

$$\lim_{n\to\infty} \|\widetilde{A}_{n,d(n)} A_n f - f\|_X = 0.$$

Regularization property

The noise model:

$$(\mathbf{\Psi}_n^\delta y)_k = (\mathbf{\Psi}_n y)_k + \delta_k, \quad |\delta_k| \le \delta, \quad y \in Y, \tag{2}$$

where $\delta > 0$ is the noise level.

Theorem: Adopt all previous assumptions.
Let $d = d(n) = d(n, f) \to \infty$ as $n \to \infty$ such that convergence holds in the noise-free setting.

If $n = n_\delta$ such that $n_\delta \to \infty$ when $\delta \to 0$ as well as

$$\delta\, \sigma\big(d(n_\delta)\big) \sqrt{n_\delta} \max_{1 \le i \le d(n_\delta)} \|\mathbf{\Phi}_{n_\delta} v_{d(n_\delta),i}\|_2 \to 0 \qquad \text{as } \delta \to 0$$

then

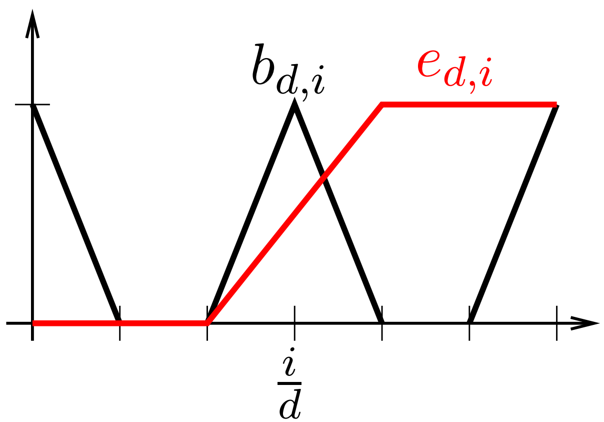
$$\limsup_{\delta \to 0} \Big\{ \big\| \widetilde{A}_{n_\delta, d(n_\delta)} \mathbf{\Psi}_{n_\delta}^\delta A f - f \big\|_X : \mathbf{\Psi}_{n_\delta}^\delta \text{ fulfills (2)} \Big\} = 0.$$

Example: the integration operator

- $X = (\mathcal{C}([0,1]), \|\cdot\|_\infty), \quad A\colon X \to X, \quad Af(x) = \int_0^x f(t)\mathrm{d}t$
- Let $\psi_{n,0} = \chi_{]0,1]}$ and $\psi_{n,k} = \chi_{[k/n,1]}$, $k = 1, \ldots, n$. Then,

$$\psi_{n,k} \in X' \quad \text{and} \quad (\mathbf{\Psi}_n g)_k = \langle \psi_{n,k}, g\rangle_{X'\times X} = g(\tfrac{k}{n}).$$

- $A_n\colon X \to \mathbb{R}^n$, $(A_n f)_k = \int_0^{\frac{k}{n}} f(t)\mathrm{d}t, \; k = 0, \ldots, n.$
- $\widetilde{A}_{n,d} w = \sum_{i=0}^{d} \langle \mathbf{\Phi}_n v_{d,i}, w\rangle_2 \, b_{d,i}, \quad v_{d,i} = -e'_{d,i},$

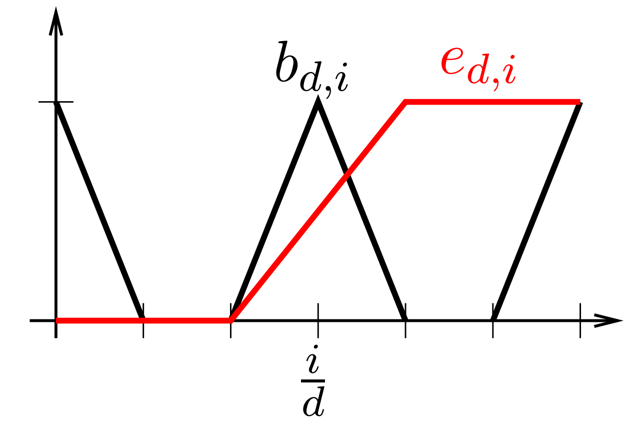


- $\mathbf{\Phi}_n\colon Y' \to \mathbb{R}^{n+1}$ is given by

$$(\mathbf{\Phi}_n v)_k = \langle v, b_{n,k}\rangle_{X'\times X}$$

Example: the integration operator (cont'd)

$$\widetilde{A}_{n,d} w = \sum_{i=0}^{d} \langle \mathbf{\Phi}_n v_{d,i}, w \rangle_2 \, b_{d,i}, \qquad v_{d,i} = -e'_{d,i},$$



If $n = d$ then $\langle \mathbf{\Phi}_n v_{d,i}, w \rangle_2$ is easily evaluated to be

$$\widetilde{A}_{n,n} w(\tfrac{k}{n}) = \langle \mathbf{\Phi}_n v_{n,k}, w \rangle_2 = n \begin{cases} w_1 - w_0 & : \ k = 0, \\ (w_{k+1} - w_{k-1})/2 & : \ k = 1, \dots, n-1, \\ w_n - w_{n-1} & : \ k = n. \end{cases}$$

Compare with the continuous setting

$$\langle v_{n,k}, Af \rangle_{X' \times X} = \frac{Af(\frac{k+1}{n}) - Af(\frac{k-1}{n})}{\frac{2}{n}}.$$

Example: the integration operator (cont'd)

We have that

$$\|\widetilde{A}_{n,d} A_n f - f\|_\infty \lesssim \|f - E_d f\|_\infty + \underbrace{\sigma(d)}_{=1} \Big(\overbrace{\rho_n}^{\lesssim n^{-1}} \underbrace{\max_{1\le i\le d} \|v_{d,i}\|_{X'}}_{\sim d} + \underbrace{\max_{1\le i\le d} \varepsilon_{d,i}}_{=0} \Big)$$

Setting $d(n) = n^{1-\gamma}$ for one $0 < \gamma < 1$ yields

$$\lim_{n\to\infty} \|f - \widetilde{A}_{n,d(n)} A_n f\|_\infty = 0 \quad \text{for any } f \in \mathcal{C}(0,1).$$

<u>Consequence</u>

We can recover about n moments of f from n observations of Af.

Remark

In a comparable Hilbert space setting we can only recover about $n^{2/3}$ moments from n observations.

Example: the integration operator (cont'd)

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$$(\boldsymbol{\Psi}_n^\delta g)_k = g(\tfrac{k}{n}) + \delta_k, \quad |\delta_k| \le \delta \tag{3}$$

Data error: $\delta\sqrt{n} \max_{1\le i\le d(n)} \|\boldsymbol{\Phi}_n v_{d(n),i}\|_2 \sim \delta\sqrt{n}\, d$

Setting $d(n) = n^{1-\gamma}$, $0 < \gamma < 1$, and $n_\delta \sim \delta^{\frac{\gamma-1}{3/2-\gamma}}$ yields

$$\lim_{\delta\to 0} \sup \Big\{ \big\| \widetilde{A}_{n_\delta, d(n_\delta)} \boldsymbol{\Psi}_{n_\delta}^\delta A f - f \big\|_\infty : \boldsymbol{\Psi}_{n_\delta}^\delta \text{ fulfills (3)} \Big\} = 0.$$

What you should recall from this talk

Approximate Inverse

- requires no geometric assumptions on Banach spaces and
- discretization effects are included into the analysis.

☞ Th. Schuster, A. Rieder, F. Schöpfer
The approximate inverse in action IV: semi-discrete equations in a Banach space setting
Inverse Problems 28 (2012)