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Seismic tomography is locally ill-posed

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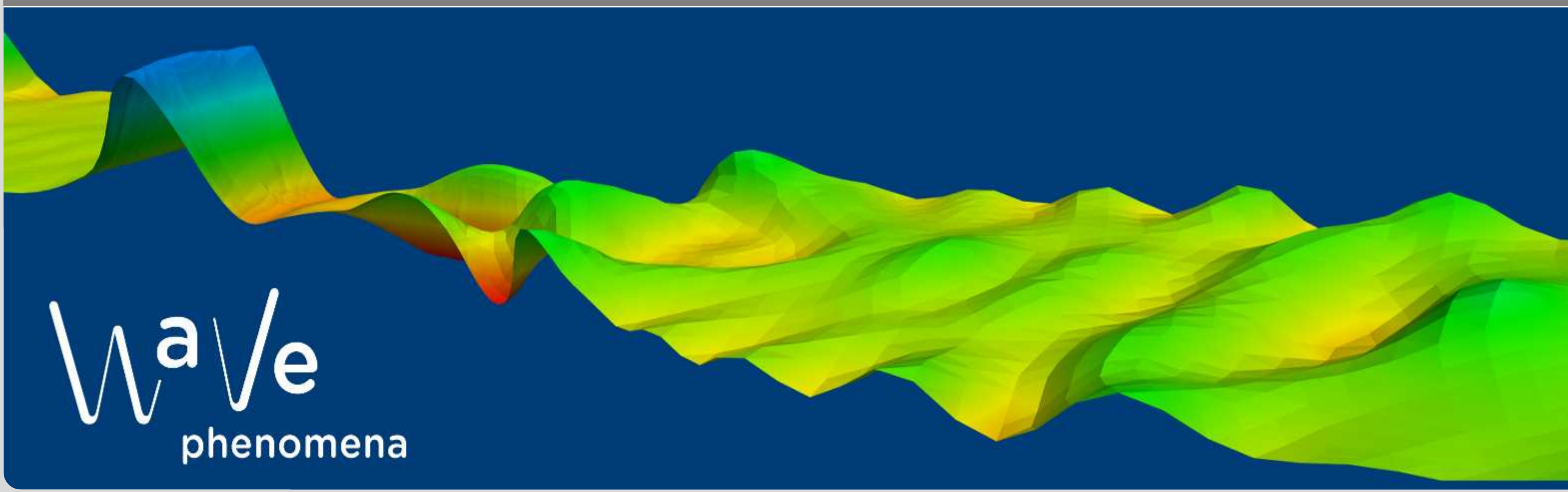


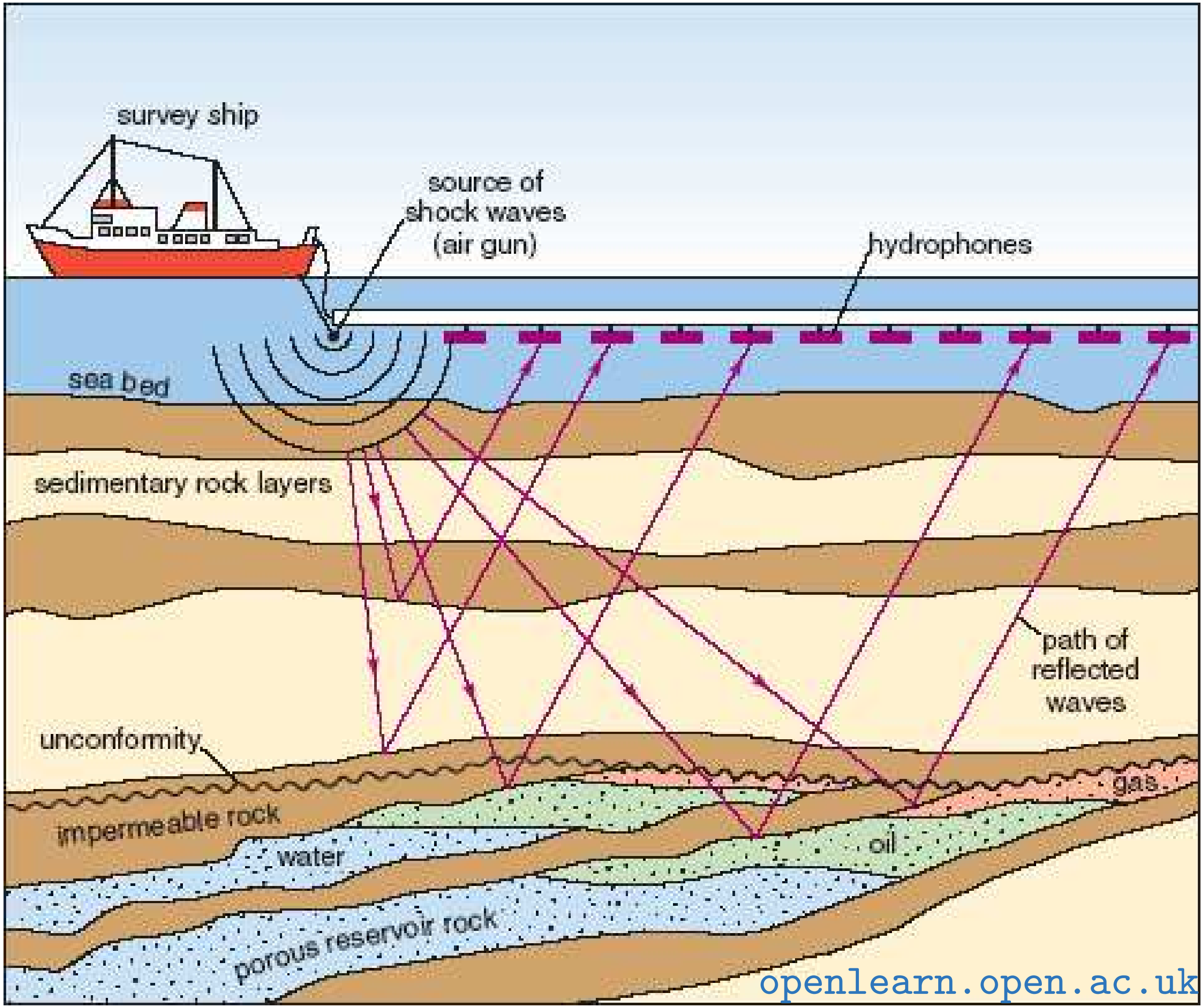
Seismic tomography: the mathematical model

The inverse problem and its ill-posedness

Final remarks

Seismic tomography: the mathematical model

Seismic tomography



Symes, *The seismic reflection inverse problem*, Inverse Problems 25, 123008 (2009)

Acoustic wave equation

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$u(t, \mathbf{x}) \in \mathbb{R}$ acoustic potential in $\mathbf{x} \in \Omega \subset \mathbb{R}^d$ at time $t \geq 0$:

$$c\,\partial_t^2 u - \nabla_{\mathbf{x}} \cdot \Big(r \nabla_{\mathbf{x}} u \Big) = f(\mathbf{x}, t), \quad u|_{\partial\Omega} = 0,$$

with initial data $u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1$ and coefficients

$$c := \frac{1}{\varrho \nu^2} \quad \text{and} \quad r := \frac{1}{\varrho}$$

where $\rho = \rho(\mathbf{x})$ mass density, $\nu = \nu(\mathbf{x})$ speed of sound.

Remark: The Dirichlet boundary restriction is quite meaningful in the framework of seismic wave propagation. According to finite wave speed and finite observation time, homogeneous boundary conditions can be assumed if Ω is chosen sufficiently large.

Acoustic wave equation: weak formulation

assumptions/notations: $c, r \in L^\infty_+(\Omega)$, $V = H^1_0(\Omega)$, $H = L^2(\Omega)$,
$u_0 \in V$, $u_1 \in H$, $f \in L^2\big((0,T), H\big) = L^2\big((0,T)\times\Omega\big)$

$$a_r : V \times V \to \mathbb{R}\,, \quad a_r(\psi,\varphi) = \int_\Omega r \nabla_{\mathbf{x}}\psi \cdot \nabla_{\mathbf{x}}\varphi \,\mathrm{d}\mathbf{x}.$$

$$X := \mathcal{C}^0([0,T], V) \cap \mathcal{C}^1([0,T], H), \quad \|u\|_X^2 := \max_{0\le t\le T} \|u(t)\|_V^2 + \max_{0\le t\le T} \|\dot u(t)\|_H^2$$

Find $u \in X$ with $u(0) = u_0$ and $\dot u(0) = u_1$ such that

$$\int_0^T \Big(a_r\big(u(t), v(t)\big) - \langle c\dot u(t), \dot v(t)\rangle_H \Big)\mathrm{d}t = \int_0^T \langle f(t), v(t)\rangle_H \,\mathrm{d}t$$

for all $v \in \mathcal{C}^\infty_0([0,T], V)$.

Properties of the weak solution

- The weak wave equation has a unique solution, which depends continously on the data and satisfies (Lions & Magenes 1972, Stolk 2000)
$$\|u(t)\|_V^2 + \|\dot{u}(t)\|_H^2 \lesssim \|u_0\|_V^2 + \|u_1\|_H^2 + \int_0^T \|f(\tau)\|_H^2 \, \mathrm{d}\tau.$$
- For almost all $s \in]0, T[$,
$$a_r\big(u(s), w\big) + \langle c\ddot{u}(s), w\rangle_{V' \times V} = \langle f(s), w\rangle_H \quad \text{for all } w \in V.$$
- $c\ddot{u} \in L^2([0,T], V')$ and $\ddot{u} \in L^2([0,T], V')$ provided $c \in W^{1,\infty}(\Omega)$.
- The weaker assumption $f \in L^2\big([0,T], V'\big)$ is not sufficient to guarantee $u \in L^2\big([0,T], V\big)$.

The inverse problem and its ill-posedness

Seismic reflection inverse problem

Seismic tomography forward operator

$$F\colon \mathsf{D}(F) \subset L^\infty(\Omega)^2 \to X, \quad (c, r) \mapsto u,$$

where $\mathsf{D}(F) = \big\{(c, r) \in L^\infty(\Omega)^2 : c(\mathbf{x}) \geq \mathrm{k}_-\,,\ r(\mathbf{x}) \geq \mathrm{k}_-\,,\ \text{a.e.}\ \big\}$

- Let $M \subset \Omega$ be the (smooth) measurement submanifold.
- Let $\Psi\colon \mathcal{C}^0([0,T], V) \to L^2([0,T] \times M)$ be the measurement operator. For instance, $\Psi\colon u \mapsto u|_M$ (trace map).

Given $w \in L^2([0,T] \times M)$ find $(c, r) \in \mathsf{D}(F)$ such that

$$\Psi F(c, r) = w.$$

Solving above problem is called full waveform inversion in seismic imaging.

Local ill-posedness in Banach spaces

$$T\colon \mathrm{D}(T) \subset X \to Y, \quad X, Y \text{ infinite dim. Banach spaces}$$

Def.: The equation $T(x) = y$ is called locally ill-posed in $x^+ \in \mathrm{D}(T)$ satisfying $T(x^+) = y$ if in any neighborhood of x^+ a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathrm{D}(T)$ can be found such that

$$\lim_{k\to\infty} \|T(x_k) - T(x^+)\|_Y = 0, \text{ however } \|x_k - x^+\|_X \not\to 0 \text{ for } k \to \infty.$$

(Hofmann 1997)

A criterion for local ill-posedness

Lemma The problem $T(x) = y$ is locally ill-posed in $x^+ \in \mathrm{D}(T)$ if

- T is compact, weak-$\star$-to-weak continuous, and
- there is $\{e_k\}_{k\in\mathbb{N}} \subset \mathrm{D}(T)$, $\|e_k\|_X = 1$, which converges weakly-$\star$ to 0 such that $\{x^+ + re_k\} \subset \mathrm{D}(F)$ for any $r \in]0, 1]$.

Proof: Define $x_k := x^+ + \rho e_k \in B_r(x^+) \cap \mathrm{D}(T)$ for any $0 < \rho < r$. We have $\|x_k - x^+\|_X = \rho$ but $x_k \overset{\star}{\rightharpoonup} x^+$.

T weak-$\star$-to-weak continuous and compact: $\|T(x_k) - T(x^+)\|_Y \to 0$. ✓

Weak-$\star$-to-weak continuity (part 1)

$$F\colon \mathrm{D}(F) \subset L^\infty(\Omega)^2 \to L^2([0,T] \times \Omega) \quad (c,r) \mapsto u,$$

$$w_k \overset{\star}{\rightharpoonup} w \text{ in } L^\infty(\Omega) \quad \Longleftrightarrow \quad \int_\Omega w_k\, v \,\mathrm{d}\mathbf{x} \xrightarrow{k\to\infty} \int_\Omega w\, v\,\mathrm{d}\mathbf{x} \;\; \forall v \in L^1(\Omega)$$

Proposition F is weak-$\star$-to-weak continous.

Proof:

- $(c_m, r_m) \overset{\star}{\rightharpoonup} (c,r) \in \mathrm{D}(F)$; $u_m = F(c_m, r_m), u = F(c,r) \in X$.
- $\{u_m\}$ and $\{\dot{u}_m\}$ are bounded in $L^2([0,T],V)$ and $L^2([0,T],H)$, resp.
- weakly convergent subsequences $\{u_{m_l}\}_{l\in\mathbb{N}}$ and $\{\dot{u}_{m_l}\}_{l\in\mathbb{N}}$ with limits η and ξ, resp.
- Observe $\dot{\eta} = \xi$.

We will show now that η solves the wave equation.

Weak-$\star$-to-weak continuity (part 2)

Let $v \in \mathcal{C}_0^\infty([0,T],V)$ and consider

$$\int_0^T \Big(a_{r_{m_l}}\big(u_{m_l}(t), v(t)\big) - \langle c_{m_l} \dot{u}_{m_l}(t), \dot{v}(t)\rangle_H \Big) \mathrm{d}t = \int_0^T \langle f(t), v(t)\rangle_H \, \mathrm{d}t.$$

We are going to show that the left hand side converges to

$$\int_0^T \Big(a_r\big(\eta(t), v(t)\big) - \langle c\dot{\eta}(t), \dot{v}(t)\rangle_H \Big) \mathrm{d}t.$$

Indeed,

$$\int_0^T \Big(a_{r_{m_l}}\big(u_{m_l}(t), v(t)\big) - a_r\big(\eta(t), v(t)\big) \Big) \mathrm{d}t =$$

$$\int_0^T a_{r_{m_l}-r}\big(u_{m_l}(t), v(t)\big) \mathrm{d}t + \underbrace{\int_0^T a_r\big(u_{m_l}(t) - \eta(t), v(t)\big) \mathrm{d}t}_{\to 0 \text{ as } u_{m_l} \rightharpoonup \eta}.$$

Weak-$\star$-to-weak continuity (part 3)

Further,

$$\Big| \int_0^T a_{r_{m_l}-r}\big(u_{m_l}(t), v(t)\big)\mathrm{d}t \Big| \le \|(r_{m_l}-r)\nabla_{\mathbf{x}} v\|_{L^2([0,T],H^d)} \|u_{m_l}\|_{L^2([0,T],V)}$$

and $|(r_{m_l}-r)\nabla_{\mathbf{x}} v|^2 \lesssim |\nabla_{\mathbf{x}} v|^2$ a.e. in $\Omega \times [0,T]$.

By the dominated convergence theorem,

$$\int_0^T \Big(a_{r_{m_l}}\big(u_{m_l}(t), v(t)\big) - a_r\big(\eta(t), v(t)\big)\Big)\mathrm{d}t \xrightarrow{l\to\infty} 0.$$

Analogously,

$$\int_0^T \Big(\big\langle c_{m_l}\dot{u}_{m_l}(t), \dot{v}(t)\big\rangle_H - \big\langle c\dot{\eta}(t), \dot{v}(t)\big\rangle_H\Big)\mathrm{d}t \xrightarrow{l\to\infty} 0.$$

Hence, η satisfies the wave equation in weak form.

Weak-$\star$-to-weak continuity (part 4)

Moreover,

$$\eta(0) = u_0 \text{ and } \dot{\eta}(0) = u_1.$$

Thus, $\eta = u$ and the whole sequence $\{u_m\}$ converges weakly to u because all convergent subsequences of $\{u_m\}$ have the limit u. ✓

Compactness (part 1)

$$F\colon \mathsf{D}(F) \subset L^\infty(\Omega)^2 \to L^2([0,T]\times\Omega) \quad (c,r)\mapsto u$$

Proposition F is compact, that is, F maps bounded sets to relatively compact ones.

Proof: Let $Q \subset \mathsf{D}(F)$ be bounded.

We show that $F(Q)$ is relatively compact in $\mathcal{C}([0,T],H)$ by the (general) theorem of Arcela-Ascoli.

By the energy estimate, for $t \in [0,T]$,

$$\{u(t) : u \in F(Q)\} \subset \{v \in V : \|\nabla_{\mathbf{x}} v\|_{L^2(\Omega)^d} \leq \hat{c}\}$$

and the latter set is relatively compact in $H = L^2(\Omega)$.

Compactness (part 2)

Furthermore, $F(Q)$ is equicontinuous because

$$\begin{aligned}\|u(t_2) - u(t_1)\|_H &= \sup_{\|\psi\|_H=1} \big\langle u(t_2) - u(t_1), \psi \big\rangle_H = \sup_{\|\psi\|_H=1} \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}s} \big\langle u(s), \psi \big\rangle_H \,\mathrm{d}s \\ &= \sup_{\|\psi\|_H=1} \int_{t_1}^{t_2} \big\langle \dot{u}(s), \psi \big\rangle_H \,\mathrm{d}s \le |t_2 - t_1| \, \|\dot{u}\|_{\mathcal{C}([0,T],H)}\end{aligned}$$

and $\|\dot{u}\|_{\mathcal{C}([0,T],H)}$ is uniformly bounded for $(c, r) \in Q$.

The continuous embedding $\mathcal{C}([0,T],H) \hookrightarrow L^2([0,T],H)$ finishes the proof. ✓

Main result

Given $w \in L^2([0,T] \times M)$ find $(c,r) \in \mathrm{D}(F)$ such that

$$\Psi F(c,r) = w.$$

Theorem The above inverse problem of seismic imaging is locally ill-posed in any point $(c_0, r_0) \in \mathrm{D}(F)$.

Proof: The assertion follows readily from the abstract criterion as soon as we have found a sequence

$$\{e_n\} \subset L^\infty(\Omega),\ e_n \geq 0 \text{ a.e.},\ \|e_n\|_\infty = 1,\ e_n \overset{\star}{\rightharpoonup} 0.$$

Let $\xi \in \Omega$ and $\rho_n \searrow 0$. Define $e_n := \chi_{B_{\rho_n}(\xi)}$.

Obviously, $e_n \geq 0$ and $\|e_n\|_\infty = 1$ and

$$\int_\omega e_n v \,\mathrm{d}\mathbf{x} \xrightarrow{n\to\infty} 0 \quad \forall v \in L^1(\Omega)$$

by the dominated convergence theorem. ✓

Final remarks

Discussion

- Ill-posedness of seismic tomography is an empirical fact known in the geophysical community for quite some time.
  - Our result shows that ill-posedness is an intrinsic feature of the mathematical model which does not originate from too few measurements.
  - Thus, regularization is not only advisable but inevitable.
- We are cautiously optimistic that our theory carries over to other boundary settings if the corresponding parameter-to-solution map can be defined in a functional analytic framework and if an energy estimate holds.