

# An abstract framework for inverse wave problems with applications

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Wave  
phenomena

# Outline

**Direct Problem**

**Inverse Problem**

**Inverse Electromagnetic Scattering**

**Summary**

▷ Direct Problem

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Inverse Problem

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Inverse Electromag-  
netic Scattering

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Summary

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# Direct Problem

# Abstract evolution equations

Consider

$$Bu'(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = u_0,$$

where

- ▶  $A: \mathcal{D}(A) \subset X \rightarrow X$   
is an (unbounded) operator on the Hilbert space  $X$ ,
- ▶  $B \in \mathcal{L}(X)$  is positive self-adjoint, and
- ▶  $f: [0, \infty[ \rightarrow X$ ,  $u_0 \in X$ .

**Note:** Neither  $A$  nor  $B$  depend on time  $t$ .

Blazek, Stolk & Symes: *A mathematical framework for inverse wave problems in heterogeneous media*, *Inverse Problems* 29 (2013).

## Example: the Maxwell system

$\mathbf{E} = \mathbf{E}(t, \mathbf{x})$  and  $\mathbf{H} = \mathbf{H}(t, \mathbf{x})$  electric and magnetic fields, resp.

$$\underbrace{\begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix}}_{= B} \begin{pmatrix} \mathbf{E}' \\ \mathbf{H}' \end{pmatrix} + \underbrace{\begin{pmatrix} \sigma I & -\text{curl}_x \\ \text{curl}_x & 0 \end{pmatrix}}_{= A} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \underbrace{\begin{pmatrix} -\mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix}}_{= f} \quad \text{in } ]0, T[ \times D$$

with bc:  $\mathbf{n} \times \mathbf{E} = 0$  on  $]0, T[ \times \partial D$

ic:  $\mathbf{E}(0, \cdot) = \mathbf{e}_0, \mathbf{H}(0, \cdot) = \mathbf{h}_0$

where  $\mathbf{J}_{e/m} = \mathbf{J}_{e/m}(t, \mathbf{x})$  current/magnetic density

$\varepsilon = \varepsilon(\mathbf{x})$  permittivity

$\mu = \mu(\mathbf{x})$  permeability

$\sigma = \sigma(\mathbf{x})$  conductivity

$D \subset \mathbb{R}^3$  bounded Lipschitz domain (or an exterior of such a domain).

## Example (continued)

- ▶  $X = L^2(D, \mathbb{R}^3) \times L^2(D, \mathbb{R}^3)$
- ▶  $\mathcal{D}(A) = H_0(\text{curl}, D) \times H(\text{curl}, D)$
- ▶ If  $0 < c \leq \varepsilon, \mu$  a.e. then  $B \in \mathcal{L}(X)$  is positive self-adjoint.

**Note:** The elastic wave equation can also be formulated in this abstract framework.

$$Bu'(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = u_0,$$

**Theorem** Let  $u_0 \in \mathcal{D}(A)$  and  $f \in W_{\text{loc}}^{1,1}([0, \infty[, X)$ . Further, let  $A$  be maximal monotone, that is,

$$\langle Av, v \rangle_X \geq 0 \quad \forall v \in \mathcal{D}(A) \quad \text{and} \quad A + I: \mathcal{D}(A) \rightarrow X \quad \text{is onto.}$$

Then, the evolution problem has a unique **classical solution**  $u \in \mathcal{C}([0, \infty[, \mathcal{D}(A)) \cap \mathcal{C}^1([0, \infty[, X)$  satisfying

$$\|u'(t)\|_X + \|u(t)\|_{\mathcal{D}(A)} \lesssim \|u_0\|_{\mathcal{D}(A)} + \|f\|_{W^{1,1}([0,t],X)} \quad \text{for all } t > 0.$$

**Proof:** 1.  $B^{-1}A$  is maximal monotone. 2. Apply the Hille-Yosida theorem to

$$u'(t) + B^{-1}Au(t) = B^{-1}f(t), \quad t > 0, \quad u(0) = u_0.$$

**Remark:**  $B^{-1}A$  generates a  $\mathcal{C}^0$  semigroup of contractions in  $(X, \langle \cdot, \cdot \rangle_B)$ .

# Weak solution: existence, uniqueness, and stability

**Corollary** Let  $u_0 \in X$ ,  $f \in L^1_{\text{loc}}([0, \infty[, X)$ ,  $A$  and  $B$  as above. Then, the evolution problem has a unique **weak** or **mild solution**  $u \in \mathcal{C}([0, \infty[, X)$ :  $u(0) = u_0$ ,

$$\frac{d}{dt} \langle Bu(t), v \rangle_X + \langle u(t), A^*v \rangle_X = \langle f(t), v \rangle_X \quad \text{for a.a. } t > 0, \quad \forall v \in \mathcal{D}(A^*).$$

Further,

$$\|u(t)\|_X \lesssim \|u_0\|_X + \|f\|_{L^1([0,t],X)} \quad \text{for all } t \geq 0.$$

**Remark:**  $A$  maximal monotone

$\implies \mathcal{D}(A)$  dense,  $A$  closed, and  $A^*$  maximal monotone as well



## Example: the Maxwell system (revisited)

$$\underbrace{\begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix}}_{= B} \begin{pmatrix} \mathbf{E}' \\ \mathbf{H}' \end{pmatrix} + \underbrace{\begin{pmatrix} \sigma I & -\text{curl}_x \\ \text{curl}_x & 0 \end{pmatrix}}_{= A} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \underbrace{\begin{pmatrix} -\mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix}}_{= f}$$

- ▶  $X = L^2(D, \mathbb{R}^3) \times L^2(D, \mathbb{R}^3)$
- ▶  $\mathcal{D}(A) = H_0(\text{curl}, D) \times H(\text{curl}, D)$

**Lemma** If  $0 \leq \sigma \in L^\infty(D)$  then  $A$  is maximal monotone.

**Proof:** For  $(\mathcal{E}, \mathcal{H})^\top \in \mathcal{D}(A)$  we have

$$\left\langle A \begin{pmatrix} \mathcal{E} \\ \mathcal{H} \end{pmatrix}, \begin{pmatrix} \mathcal{E} \\ \mathcal{H} \end{pmatrix} \right\rangle_X = \int_D [(\sigma \mathcal{E} - \text{curl}_x \mathcal{H}) \cdot \mathcal{E} + \text{curl}_x \mathcal{E} \cdot \mathcal{H}] dx = \int_D \sigma |\mathcal{E}|^2 dx \geq 0$$

by Green's theorem (no boundary term appears as  $\mathcal{E} \in H_0(\text{curl}, D)$ ).

Direct Problem

▷ Inverse Problem

Inverse Electromag-  
netic Scattering

Summary

# Inverse Problem

# Parameter-to-solution map

For  $u_0 \in \mathcal{D}(A)$ ,  $f \in W_{\text{loc}}^{1,1}([0, \infty[, X)$ ,  $A$  maximal monotone, consider

$$F: \mathcal{D}(F) \subset \mathcal{L}(X) \rightarrow \mathcal{C}([0, T], X), \quad B \mapsto u$$

where  $T > 0$ ,

$$\mathcal{D}(F) \subset \mathcal{B} = \{B \in \mathcal{L}(X) : B = B^*, \beta \|v\|_X^2 \leq \langle Bv, v \rangle_X \forall v \in X\}, \quad \beta > 0,$$

and  $u$  solves

$$Bu'(t) + Au(t) = f(t), \quad t \in ]0, T[, \quad u(0) = u_0.$$

# Local ill-posedness of parameter identification

**Theorem** The inverse problem

$$F(\cdot) = u$$

is **locally ill-posed** at any  $\hat{B} \in \mathcal{D}(F)$  satisfying  $F(\hat{B}) = u$  if there are an  $\hat{r} > 0$  and a family  $\{E_k\} \subset \mathcal{L}(X)$  of positive semi-definite operators s.t.

$$\forall r \in ]0, \hat{r}]: \hat{B} + rE_k \in \mathcal{D}(F), \quad \|E_k\| \sim 1, \quad E_k \xrightarrow{k \rightarrow \infty} 0 \text{ pointwise.}$$

**Proof:** Set  $B_k := \hat{B} + rE_k$ . Since

$$\|B_k - \hat{B}\| \sim r$$

it remains to show that

$$F(B_k) \xrightarrow{k \rightarrow \infty} F(\hat{B}) \text{ in } \mathcal{C}([0, T], X).$$

# Fréchet differentiability

**Theorem**  $F: \mathcal{B} \subset \mathcal{L}(X) \rightarrow \mathcal{C}([0, T], X)$ ,  $B \mapsto u$ , is Fréchet differentiable at any  $B \in \text{int}(\mathcal{B})$  with derivative

$$F'(B)H = u_H|_{[0, T]}$$

where  $u_H$  is the weak solution of

$$Bu'_H(t) + Au_H(t) = -Hu'(t), \quad t > 0, \quad u_H(0) = 0,$$

with  $u = F(B)$ , i.e., classical solution of  $Bu'(t) + Au(t) = f(t)$ ,  $u(0) = u_0$ .

**Remark:** Fréchet differentiability of

$$F: \mathcal{B} \subset \mathcal{L}(X) \rightarrow \mathcal{C}^1([0, T], X) \cap \mathcal{C}([0, T], \mathcal{D}(A))$$

requires stronger regularity assumptions on  $f$  and  $u_0$ , namely

$$f \in W^{3,1}([0, T], X), \quad v := B^{-1}(Au_0 - f(0)) \in \mathcal{D}(A), \quad B^{-1}(Av - f'(0)) \in \mathcal{D}(A).$$

# Adjoint of the Fréchet derivative

Here,  $F'(B): \mathcal{L}(X) \rightarrow L^2([0, T], X)$  linear and bounded. Then,

$$F'(B)^*: L^2([0, T], X) \rightarrow \mathcal{L}(X)^*.$$

**Theorem** We have that

$$[F'(B)^*g]H = \int_0^T \langle Hu'(t), w(t) \rangle_X dt$$

where  $w$  is the weak solution of the backwards equation

$$Bw'(t) - A^*w(t) = g(t), \quad t \in ]0, T[, \quad w(T) = 0,$$

and  $u = F(B)$ , i.e., classical solution of  $Bu'(t) + Au(t) = f(t)$ ,  $u(0) = u_0$ .

**Remark:** Set  $\tilde{w}(t) = w(T - t)$ . Then, the backwards eqn. is equivalent to

$$B\tilde{w}'(t) + A^*\tilde{w}(t) = -g(T - t), \quad t \in ]0, T[, \quad \tilde{w}(0) = 0.$$

Direct Problem

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Inverse Problem

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Inverse Elec-  
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Summary

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# Inverse Electromagnetic Scattering

# Parameter-to-solution map

$$\begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix} \begin{pmatrix} \mathbf{E}' \\ \mathbf{H}' \end{pmatrix} + \begin{pmatrix} \sigma I & -\text{curl}_x \\ \text{curl}_x & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix}, \quad \begin{pmatrix} \mathbf{E}(0) \\ \mathbf{H}(0) \end{pmatrix} = \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{h}_0 \end{pmatrix}$$

- ▶  $X = L^2(D, \mathbb{R}^3) \times L^2(D, \mathbb{R}^3)$
- ▶  $\mathcal{D}(A) = H_0(\text{curl}, D) \times H(\text{curl}, D)$
- ▶  $(\mathbf{e}_0, \mathbf{h}_0)^\top \in \mathcal{D}(A)$  and  $(\mathbf{J}_e, \mathbf{J}_m)^\top \in W^{1,1}([0, T], X)$  for  $T > 0$
- ▶  $\mathcal{P} := \{(\varepsilon, \mu)^\top \in L^\infty(D)^2 : 0 < c \leq \varepsilon, \mu \text{ a.e.}\}$

Then, the parameter-to-solution map is well defined:

$$\Phi: \mathcal{P} \subset L^\infty(D)^2 \rightarrow \mathcal{C}([0, T], X), \quad (\varepsilon, \mu)^\top \mapsto (\mathbf{E}, \mathbf{H})^\top.$$



# Factorization of the parameter-to-solution map

We factorize

$$\Phi = F \circ V$$

where

$$F: \mathcal{B} \subset \mathcal{L}(X) \rightarrow \mathcal{C}([0, T], X), \quad B \mapsto (\mathbf{E}, \mathbf{H})^\top,$$

and

$$V: \mathcal{P} \subset L^\infty(D)^2 \rightarrow \mathcal{B}, \quad (\varepsilon, \mu)^\top \mapsto \begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix}.$$

Note that  $F$  is the mapping considered in the abstract theory.

# Inverse electromagnetic scattering is locally ill-posed

**Theorem** The inverse problem

$$\Phi(\varepsilon, \mu) = (\mathbf{E}, \mathbf{H})^\top$$

is locally ill-posed at any interior point of  $\mathcal{P}$ .

**Proof:** Let

$$e_k := \chi_{B_{1/k}}(\xi) \quad \text{for one fixed } \xi \in D$$

and define

$$E_k := \begin{pmatrix} e_k I & 0 \\ 0 & e_k I \end{pmatrix} = V(e_k, e_k).$$

Then,  $\|E_k\| = 1$ ,  $E_k \xrightarrow{k \rightarrow \infty} 0$  pointwise, and, for any  $r > 0$ ,

$$\Phi(\varepsilon + r e_k, \mu + r e_k) = F(V(\varepsilon, \mu) + r E_k) \xrightarrow{k \rightarrow \infty} F(V(\varepsilon, \mu)) = \Phi(\varepsilon, \mu)$$

but

$$\varepsilon + r e_k \not\rightarrow \varepsilon \quad \text{as well as} \quad \mu + r e_k \not\rightarrow \mu \quad (\text{due to } \|e_k\|_\infty = 1).$$

# Fréchet derivative of the electromagnetic forward map

**Theorem** The parameter-to-solution map

$$\Phi: \mathcal{P} \subset L^\infty(D)^2 \rightarrow \mathcal{C}([0, T], X)$$

is Fréchet differentiable at  $(\varepsilon, \mu)^\top$ . In fact,

$$\Phi'(\varepsilon, \mu) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{pmatrix} \overline{\mathbf{E}} \\ \overline{\mathbf{H}} \end{pmatrix}$$

where  $(\overline{\mathbf{E}}, \overline{\mathbf{H}})^\top \in \mathcal{C}([0, T], X)$  is the weak solution of

$$\begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix} \begin{pmatrix} \overline{\mathbf{E}}' \\ \overline{\mathbf{H}}' \end{pmatrix} + \begin{pmatrix} \sigma I & -\text{curl}_x \\ \text{curl}_x & 0 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{E}} \\ \overline{\mathbf{H}} \end{pmatrix} = \begin{pmatrix} -h_1 \mathbf{E}' \\ -h_2 \mathbf{H}' \end{pmatrix},$$

in  $[0, T] \times D$  with  $\overline{\mathbf{E}}(0) = \overline{\mathbf{H}}(0) = \mathbf{0}$ . Here,  $(\mathbf{E}, \mathbf{H})^\top = \Phi(\varepsilon, \mu)$ .

**Proof:**  $\Phi'(\varepsilon, \mu) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = F'(V(\varepsilon, \mu))V'(\varepsilon, \mu) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = F'(V(\varepsilon, \mu))V(h_1, h_2)$

# Adjoint of the Fréchet derivative

**Theorem** The dual operator  $\Phi'(\varepsilon, \mu)^* : L^2([0, T], X) \rightarrow (L^\infty(D)^*)^2$  is given by

$$\Phi'(\varepsilon, \mu)^* \begin{pmatrix} \mathbf{g}_E \\ \mathbf{g}_H \end{pmatrix} = \begin{pmatrix} \int_0^T \mathbf{E}'(t, \cdot) \cdot \bar{\mathbf{E}}(t, \cdot) dt \\ \int_0^T \mathbf{H}'(t, \cdot) \cdot \bar{\mathbf{H}}(t, \cdot) dt \end{pmatrix} \in L^1(D)^2$$

where  $(\bar{\mathbf{E}}, \bar{\mathbf{H}})^\top \in \mathcal{C}([0, T], X)$  uniquely solves

$$\begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix} \begin{pmatrix} \bar{\mathbf{E}}' \\ \bar{\mathbf{H}}' \end{pmatrix} + \begin{pmatrix} -\sigma I & -\text{curl}_x \\ \text{curl}_x & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_E \\ \mathbf{g}_H \end{pmatrix}$$

with  $(\bar{\mathbf{E}}(T), \bar{\mathbf{H}}(T))^\top = (0, 0)^\top$ .

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▷ Summary

# Summary

# Things to remember

$$Bu'(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = u_0$$

$B \in \mathcal{L}(X)$  pos. self-adjoint;  $A: \mathcal{D}(A) \subset X \rightarrow X$  maximal monotone;  
 $X$  Hilbert space

We have investigated  $F: B \mapsto u$  and shown the

- ▶ Fréchet-differentiability of  $F$ ,
- ▶ Local ill-posedness of  $F(\cdot) = u$ ,

and applied our abstract findings to

- ▶ Maxwell's system (inverse electromagnetic scattering).

**Final note:** The abstract theory applies to the elastic wave equation (seismic tomography) as well.



A. Kirsch, A. Rieder: *Inverse problems for abstract evolution equations with applications in electrodynamics and elasticity*, Inverse Problems (2016), to appear