Approximate inverse for the common offset acquisition geometry in 2D seismic imaging

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Organization of the material

The Problem
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The Experiments
The Future
The Problem
An inverse problem for the acoustic wave equation

\[ u(t; x, x_s) \] acoustic potential in \( x \in \mathbb{R}^2 \) at time \( t \geq 0 \)

\[ \frac{1}{\nu^2} \partial_t^2 u - \Delta_x u = \delta(x - x_s)\delta(t) \]

\( \nu = \nu(x) \) speed of sound, \( x_s \) excitation (source) point.

**Seismic imaging**

Recover \( \nu \) from the backscattered (reflected) fields

\[ u(t; x_r, x_s), \quad t \in [0, T_{\text{max}}], \quad (x_r, x_s) \in \mathcal{R} \times \mathcal{S} \]

where

\( \mathcal{S}/\mathcal{R} \) sets of source/receiver points, and

\( T_{\text{max}} \) observation period.
Generalized Radon transform

Consider the ansatz
\[ \frac{1}{\nu^2(x)} = \frac{1 + n(x)}{c^2(x)}, \]
\( c = c(x) \) smooth and known background velocity.

Determine \( n \) from
\[ Fw(T; x_r, x_s) = \int_0^T (u - \tilde{u})(t; x_r, x_s) \, dt \]
with the generalized Radon transform
\[ Fw(T; x_r, x_s) = \int \frac{w(x)}{c^2(x)} a(x, x_s) a(x, x_r) \delta(T - \tau(x, x_s) - \tau(x, x_r)) \, dx \]
which integrates \( w \) over reflection isochrones: \( T = \tau(\cdot, x_s) + \tau(\cdot, x_r) \).

Travel-time \( \tau \) and amplitude \( a \) can be computed from
\[ |\nabla_x \tau| = c^{-1} \quad \text{and} \quad \text{div}(a^2 \nabla_x \tau) = 0. \]
Historical note: Kirchhoff migration

- Since the 1950’s Kirchhoff migration is the standard technique to approximately solve the integral equation.

- Beylkin (1984, 1985) showed that there is a convolution type operator $K$ and a dual transform $F^\#$ such that

\[ F^\# K F = I + \Psi \]

where $\Psi$ is compact. Further, Kirchhoff migration is the direct application of $F^\# K$ to the measured data.
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We advocate a different approach which we think

- is more flexible,
- allows a better control of the involved parameters, and
- gives a better understanding of the propagation of singularities

which hopefully results in better reconstructions.
The elliptic Radon transform in 2D

- background velocity $c = 1$: $\tau(x, y) = |x - y|$ and $a(x, y) = 1/|x - y|$
- $n$ is compactly supported in the lower half space $x_2 > 0$ ($x_2 > 0$ points downwards),

common offset scanning geometry:

$$x_s(s) = (s - \alpha, 0)^\top \quad \text{and} \quad x_r(s) = (s + \alpha, 0)^\top$$

where $\alpha > 0$ is the common offset.
The elliptic Radon transform (continued)

In this situation the generalized Radon transform integrates over ellipses and may be written as

\[ Fw(s, t) = \int A(s, x)w(x)\delta(t - \varphi(s, x))dx, \quad t > 2\alpha, \]

with

\[ \varphi(s, x) := |x_s(s) - x| + |x_r(s) - x| \]

and

\[ A(s, x) = \frac{1}{|x_s(s) - x| |x_r(s) - x|}. \]
The imaging operator

As an inversion formula for $F$ is unknown we define the reconstruction operator

$$\Lambda = \Delta F^* \Phi F$$

where

- $\Phi = \Phi(s, t)$ is a smooth compactly supported cutoff function,
- $F^*$ is a formal (weighted) $L^2$-adjoint of $F$, and
- $\Delta$ is the Laplacian.

From the elliptic means $g = F n$ we can recover

$$\Lambda n = \Delta F^* \Phi g.$$
Why this choice of $\Lambda$?

- $\Lambda$ is a $\Psi do$ of order 1 and $\Lambda n$ emphasizes singularities (e.g., jumps along curves) of $n$ which are tangent to ellipses being integrated over.
  (follows from results by Guillemin & Sternberg, 1977, and by Krishnan et al., 2012)
Inversion scheme: approximate inverse

Instead of $\Lambda n(p)$ we try to compute

$$\Lambda_{\gamma} n(p) := \langle \Lambda n, e_{p,\gamma} \rangle_{L^2(\mathbb{R}^2)} = \Lambda n \ast e_{0,\gamma}(p)$$

where $e_{p,\gamma}, \gamma > 0$, is a mollifier:

$$\supp e_{p,\gamma} = \overline{B_{\gamma}(p)}, \quad \int e_{p,\gamma}(x) dx = 1, \quad e_{p,\gamma} \xrightarrow{\gamma \to 0} \delta(\cdot - p).$$

We use

$$e_{p,\gamma,k}(x) = C_{k,\gamma} \begin{cases} (\gamma^2 - \Theta^2)^k : \Theta < \gamma, \\ 0 : \Theta \geq \gamma, \end{cases} \quad \Theta = |x - p|,$$

with $k > 0$ and

$$C_{k,\gamma} = \frac{k + 1}{\pi \gamma^{2(k+1)}}.$$
Inversion scheme: reconstruction kernel

**Lemma:** For $k \geq 3$ we have that

$$\Lambda_{\gamma} n(p) = \langle \Phi F n, \psi_{p,\gamma,k} \rangle_{L^2(\mathbb{R} \times [2\alpha,\infty],t^2 \, dt \, ds)}$$

with the **reconstruction kernel**

$$\psi_{p,\gamma,k}(s,t) = 4k \, C_{k,\gamma} \left( (k-1) \, F\left( | \cdot - p |^2 \tilde{e}_{p,\gamma,k-2} \right)(s,t) - F\tilde{e}_{p,\gamma,k-1}(s,t) \right)$$

with $\tilde{e}_{p,\gamma,k} = e_{p,\gamma,k}/C_{k,\gamma}$.

**Proof:** By duality, $\Lambda_{\gamma} n(p) = \langle \Delta F^* \Phi F n, e_{p,\gamma,k} \rangle = \langle \Phi F n, \psi_{p,\gamma,k} \rangle$ with

$$\psi_{p,\gamma,k} = F\Delta e_{p,\gamma,k} = C_{k,\gamma} F\Delta\tilde{e}_{p,\gamma,k}$$

and $\Delta\tilde{e}_{p,\gamma,k} = 4k(k-1) | \cdot - p |^2 \tilde{e}_{p,\gamma,k-2} - 4k \tilde{e}_{p,\gamma,k-1}$ yields the result. $\checkmark$
Plotting the kernel

\[ \psi_{p, \gamma, 3}, \alpha = 1.00, \gamma = 0.80, p = (0.00, 3.00) \]

Left:

- \( t \) vs. midpoint \( s \)
- Travel time vs. diameter

Right:

- Travel time (diameter) vs. \( \psi_{p, \gamma, 3}(0, :) \), \( \alpha = 1.00, \gamma = 0.80, p = (0.00, 3.00) \)
The Experiments
Discretization

We compute

\[ \Lambda_\gamma n(p) = \langle \Phi F n, \psi_{p,\gamma,3} \rangle_{L^2(\mathbb{R} \times ]2\alpha, \infty[)} \]

from the discrete data

\[ g(i, j) = \Phi(s_i, t_j) F n(s_i, t_j), \quad i = 1, \ldots, N_s, \quad j = 1, \ldots, N_t, \]

where

\[ \{s_i\} \subset [-s_{\text{max}}, s_{\text{max}}] \quad \text{and} \quad \{t_j\} \subset [t_{\text{min}}, t_{\text{max}}], \quad t_{\text{min}} > 2\alpha, \]

are uniformly distributed with step sizes \( h_s \) and \( h_t \), respectively.

\[ \Lambda_\gamma n(p) \approx \tilde{\Lambda}_\gamma n(p) := h_s h_t \sum_{i=1}^{N_s} \sum_{t_j \in \mathcal{J}_i(p)} g(i, j) \psi_{p,\gamma,3}(s_i, t_j) t_j^2 \]

with \( |\mathcal{J}_i(p)| \sim \gamma \).
The phantom $n$ and its transform $\Phi F n$

\[ N_s = N_t = 600 \]
Reconstructed images $\tilde{\Lambda}_{\gamma n}$

\[ \alpha = 2.00, \gamma = 0.20, \ s_{\min} = -15.00, \ t_{\max} = 34.90 \]

\[ \alpha = 5.00, \gamma = 0.20, \ s_{\min} = -15.00, \ t_{\max} = 40.50 \]
Reconstructed images $\tilde{\Lambda}_{\gamma n}$: limited data

$\alpha = 5.00, \gamma = 0.20, s_{\min} = -7.50, t_{\max} = 25.50$

$s_i \in [-7.5, 7.5]$
Data from the wave equation

\[
Fw(T; x_r, x_s) = \int_0^T (u - \tilde{u})(t; x_r, x_s) \, dt
\]

- \[[0.1, 1] \times [0.1, 0.8]\] with absorbing bc using PML. Step size 0.01.
- 17 source/receiver pairs, \(\alpha = 0.05\), positioned at \((s \pm \alpha, 0.1)\), \(s \in \{0.15 + 0.05i : i = 0, \ldots, 16\}\), to record \(u\) at the receivers.
- Temporal source signal: scaled Gaussian.
- \(\tilde{u}\) was computed with constant sound speed \(c = 1\).
Wavefields

Sine profile

Cosine profile

PySIT — Seismic Imaging Toolbox for Python
by L. Demanet & R. Hewitt
Preprocessed seismograms

\[ y(s, t) = \int_0^T (u - \tilde{u})(t; x_r(s), x_s(s)) \, dt \]
Reconstructed images $\tilde{\Lambda}_{0.06n}$
The Future
Next steps

- Generalization to 3D
- Symbol calculation for $\Lambda$
  \[
  \Lambda n(y) = \int \int n(x)\sigma(x, y, p)e^{-ip(x-y)}dx dp
  \]
- New reconstruction kernels
- Non-constant background velocity
Next steps

- Generalization to 3D
- Symbol calculation for $\Lambda$

$$\Lambda n(y) = \int \int n(x) \sigma(x, y, p) e^{-ip(x-y)} dx dp$$

- New reconstruction kernels
- Non-constant background velocity

Thank you for your attention!
Why this choice of $\Lambda$?

$\Lambda$ is a $\Psi$do of order 1 and $\Lambda n$ emphasizes singularities (e.g., jumps along curves) of $n$ which are tangent to ellipses being integrated over.

Proof:

- Under the Bolker assumption any hypersurface Radon transform $R$ on $\mathbb{R}^d$ and its (formal, smoothly weighted) $L^2$-adjoint $R^*$ are FIOs of order $-(d - 1)/2$. (Guillemin & Sternberg, 1977)
- If they can be composed, then $R^* R$ is a $\Psi$do.
- Our $F$ on $\mathbb{R}^2$ satisfies the Bolker assumption (Krishnan et al., 2012), that is, $F^* \Phi F$ is of order $-1$. 
Reconstructed images $\tilde{\Lambda}_\gamma n$: erroneous offset
Computing the kernel

Let $\chi$ be the indicator function of $B_r(p)$ which is in the lower half-space. To evaluate

$$F\chi(s,t) = \int A(s, x) \chi(x) \delta(t - \varphi(s, x)) \, dx, \quad t > 2\alpha,$$

we transform the integral by elliptic coordinates $x(s, t, \phi) = (x_1, x_2)^\top$,

$$x_1 = s + \frac{t}{2} \cos \phi \quad \text{and} \quad x_2 = \sqrt{\frac{t^2}{4} - \alpha^2} \sin \phi.$$

Note: $E(s, t) = \{x(s, t, \phi) : \phi \in [0, 2\pi]\}$ ellipse wrt $x_s(s), x_r(s)$, and $t$.

Thus,

$$F\chi(s, t) = \frac{1}{\sqrt{t^2 - 4\alpha^2}} \int_0^\pi \chi(x(s, t, \phi)) \, d\phi.$$
Computing the kernel (continued)

To evaluate $F_{\chi}(s, t)$ further we provide the following quantities

$$T_{-/+} = T_{-/+}(s, r, p) = \min / \max \{ \varphi(s, x) : x \in \partial B_r(p) \}.$$ 

$$E(s, t) \cap B_r(p) \neq \emptyset \iff T_- < t < T_+$$

For $t \in [T_-, T_+]$:

$$E(s, t) \cap B_r(p) = \{ x(s, t, \phi) : \phi \in [\phi_1, \phi_2] \}$$

$$F_{\chi}(s, t) = \begin{cases} 
0 & : t \notin ]T_-, T_+[ \\
\frac{\phi_2 - \phi_1}{\sqrt{t^2 - 4\alpha^2}} & : t \in ]T_-, T_+[ 
\end{cases}$$
Computing the kernel (continued)

Remaining tasks: Compute $T_{-/+}$, $\phi_{1/2}$.

$$T_{-/+} = \min / \max \left\{ \tilde{\varphi}(\vartheta) : \vartheta \in [0, 2\pi[ \right\}$$

where

$$\tilde{\varphi}(\vartheta) := \varphi(s, p + r(\cos \vartheta, \sin \vartheta)^\top).$$

- $\tilde{\varphi}$ attains exactly one minimum and one maximum in $[0, 2\pi[.$
- As both extrema are clearly separated, we can apply Newton’s method to get the two zeros of $\tilde{\varphi}'$. 
Computing the kernel (continued)

- Having $T_{\pm}$ we solve

$$r^2 = |p - x(s, t, \phi)|^2 \quad \text{for } \phi.$$  

For $t \in ]T_-, T_+[,$ $s \in \mathbb{R}$ we have exactly the two solutions $\phi_1$ and $\phi_2$.

- We substitute

\[
\begin{align*}
    z &= \cos \phi, \\
    b &= (s - p_1) t, \\
    c &= (p_1 - s)^2 + p_2^2 + \frac{t^2}{4} - \alpha^2 - r^2, \\
    d &= \sqrt{t^2 - 4\alpha^2 p_2},
\end{align*}
\]

leading to the equation

$$d \sqrt{1 - z^2} = c + b z + \alpha^2 z^2,$$

which has exactly two solutions $-1 \leq z_2 < z_1 \leq 1$.

- By Newton’s method again,

$$\phi_i = \arccos z_i, \quad i = 1, 2.$$
The kernel $\psi_{p,\gamma,k} = F\Delta e_{p,\gamma,k}$ can be computed just as $F\chi$.

Indeed, let $k = 3$, then

$$\Delta e_{p,\gamma,3}(x) = C_{3,\gamma}(-36|x-p|^4 + 48\gamma^2|x-p|^2 - 12\gamma^4)\chi_{B_\gamma(p)}(x).$$

Now $F$ can be applied to each of the components of $\Delta e_{p,\gamma,3}$, e.g.,

$$F(|\cdot - p|^4\chi_{B_\gamma(p)})(s,t) = \begin{cases} 0 & : t \notin ]T_-,T_+[ , \\ \frac{1}{\sqrt{t^2 - 4\alpha^2}} \int_{\phi_1}^{\phi_2} |x(s,t,\phi) - p|^4 \, d\phi & : t \in ]T_-,T_+[ . \end{cases}$$

Here,

$$|x(s,t,\phi) - p|^4 = \left( (s - p_1 + \frac{t}{2}\cos\phi)^2 + \left(\sqrt{\frac{t^2}{4} - \alpha^2 \sin\phi - p_2}\right)^2 \right)^2$$

is a trigonometric polynomial which can be integrated analytically.