Worksheet 12
Advanced Mathematics I for Mechanical Engineering

Problem 56: Show that for \( x \in [-1, 1] \)
\[
-1 + x - \frac{x^2 - 1}{x^2 + 1} \leq \arctan(x) \leq 1 + x + \frac{x^2 - 1}{x^2 + 1}.
\]
Hint: Analyze the extrema of the terms’ differences.

Problem 57: What is the domain of
\[
f(x) = \ln(x + 1) - \frac{x^2}{2 + 2x},
\]
for \( x \in [0, 2] \)?

Problem 58: Compute the following limits:
\[
\begin{align*}
(a) & \quad \lim_{x \to 0} \left( \frac{1}{x^3 + ax^2 + x} - \frac{1}{\sin x} \right), \\
(b) & \quad \lim_{x \to 0} \left( \frac{1}{x} \right)^{\tan x}, \\
(c) & \quad \lim_{x \to 0} (\cos x)^{1/x^2}.
\end{align*}
\]
The coefficient \( a \in \mathbb{R} \) in part (a) is a parameter.

Problem 59: Compute all derivatives \( f^{(n)}(x) \), \( n = 0, 1, 2, \ldots \) of the function
\[
f: \begin{cases} 
(-1, 1) & \longrightarrow \mathbb{R} \\
x & \mapsto \ln \frac{1 + x}{1 - x}
\end{cases}
\]
and determine the Taylor series expansion of \( f \) at \( x_0 = 0 \). For which \( x \in \mathbb{R} \) does this power series converge?

Problem 60: Using the Taylor’s formula, show that
\[
\begin{align*}
(a) & \quad (1 + x)^n \geq 1 + nx, \\
(b) & \quad (1 + x)^{\frac{1}{n}} \leq 1 + \frac{x}{n}.
\end{align*}
\]
for \( n \in \mathbb{N} \) and \( 1 + x > 0 \).

Problem 61: Compute the Taylor polynomial of degree \( n = 5 \) for \( f(x) = \sin^2 x \). Use it to approximate \( \sin^2 \frac{1}{10} \) and show that the error is less than \( 10^{-6} \).

Due date: Tuesday, February 7, 2006, 8:00 am (in the tutorial)
Problem T45: Show that
\[
\arcsin(x) \leq \frac{\pi}{2} + 2x\sqrt{1-x^2},
\]
for \(x \in [-1, 1]\).

Problem T46:
(a) Compute the following limits:

\[
\text{(i) } \lim_{x \to \infty} \frac{x^2 e^x}{(e^x - 1)^2}, \quad \text{(ii) } \lim_{x \to 0} \frac{\sin^2 x}{x \sin(x^2)}.
\]

(b) Determine the constant \(c \in \mathbb{R}\), so that the function

\[
f(x) = \begin{cases} 
  c, & x = 1, \\
  \frac{2\ln x - x}{\ln x}, & x \neq 1,
\end{cases}
\]
is continuous in \(\mathbb{R}_{>0}\).

Problem T47: Use Taylor’s formula of second degree with expansion point \(x_0 = 0\) and the remainder term in Lagrange form to prove the estimate

\[
x - \frac{1}{2} x^2 + \frac{1}{3} \left( \frac{x}{1+x} \right)^3 \leq \ln(1 + x) \leq x - \frac{1}{2} x^2 + \frac{1}{3} x^3, \quad 0 \leq x < \infty.
\]