Worksheet 2
Advanced Mathematics I for Mechanical Engineering

Problem 6: Using mathematical induction over $n \in \mathbb{N}$, show that:

(a) \[ \sum_{k=1}^{n} \frac{1}{n+k} \geq \frac{1}{2}, \]

(b) \[ \sum_{k=1}^{n} \frac{1}{k} \geq 1 + \frac{n}{2}. \]

Problem 7: Given the complex numbers

\[ z_1 = 1 + i, \quad z_2 = 2 - 3i, \quad z_3 = \sqrt{3} + i, \]

compute

(a) the real and the imaginary part of the numbers $\overline{z_j}$, $-z_j$, $z_j \overline{z_j}$, $\frac{1}{z_j}$, $z_j - \overline{z_j}$ and $|z_j|$, $j = 1, 2$, as well as

\[ \frac{z_1}{z_1 + z_2}, \quad \text{and} \quad z_1^2 z_2. \]

(b) $z_3$’s representation in polar coordinates $(r, \varphi)$.

Problem 8: Determine the real and the imaginary part of all $\omega \in \mathbb{C}$, which satisfy the following relation:

\[ \omega^2 = \frac{3}{1-3i} - \frac{1}{3+i}, \]

(a) by means of the substitution $\omega = x + iy$;

(b) using polar coordinates.

Problem 9: Given the following subsets of $\mathbb{C}$

\[ M_1 := \left\{ \frac{t^2 + 3 + i2t}{t^2 + 1} : t \in \mathbb{R} \right\}, \quad M_2 := \{3 + is : s \in \mathbb{R}\}, \]

show that

(a) $|z - 2| = 1$ for all $z \in M_1$.

(b) $M_2 = \left\{ \frac{2z}{z-1} : z \in M_1 \right\}$.

Without proof: Which geometrical figures in the complex plane do $M_1$ and $M_2$ represent?

Problem 10: Prove by induction over $n \in \mathbb{Z}_{\geq 0}$:

\[ \sum_{k=0}^{2n} i^k k = \begin{cases} n(1 - i), & n \text{ even} \\ -(n + 1) + ni, & n \text{ odd} \end{cases} \]

Due date: Tuesday, November 15, 2005, 8:00 am (in the tutorial)
Problem T5: Given the complex number \(c\), compute \(-c\), \(\overline{c}\), \(-c + \overline{c}\), \(c - \overline{c}\), \(|c|\) and \(\text{Arg } c\):

(a) \(c_1 = 1 + i\),
(b) \(c_2 = 4 - 3i\),
(c) \(c_3 = -5 + 12i\).

Problem T6: Let \(z_1 = 1 + i\), \(z_2 = 1 - i\), \(z_3 = 4 - 3i\). Compute

\[
\frac{z_1}{z_2}, \quad z_1 \cdot z_3, \quad z_1 - z_2, \quad \frac{z_3}{z_1}, \quad \frac{z_2}{z_1}, \quad \frac{z_1 \cdot z_2}{z_3}, \quad \frac{z_3}{z_1 + z_2}, \quad \frac{z_3}{z_1 - z_2}.
\]

Problem T7: Which complex numbers satisfy the conditions

(a) \((\text{Im}(2z + i))^2 - 1 \leq 4|z|^2 - 8\text{Re}(z) < -(z - \overline{z})^2)\),

(b) \(z^4 + (2i + 2)z^2 + 4i = 0\)?

Sketch the set of solutions of part a) in the complex plane.

Problem T8: Show that

\[
\sum_{j=1}^{n} \left[ \cos \left( \frac{2\pi j}{n} \right) + i \sin \left( \frac{2\pi j}{n} \right) \right] = 0.
\]

for all \(n \in \mathbb{N}\).

Illustrate the statement in the case \(n = 3\).

Hint: Using mathematical induction, show that for \(j \in \mathbb{N}\), \(\alpha \in \mathbb{R}\)

\[
\cos (\alpha j) + i \sin (\alpha j) = (\cos \alpha + i \sin \alpha)^j,
\]

and then apply the formula for the sum of a geometric series.

Tutorial: Tuesday, November 8, 2005, 8:00 am (room 203 in the ID).
Looking forward to meeting you!