Problem 21:  Let the sequence \((a_n)\) be defined recursively by an initial value \(a_1 \in [0, 2]\) and the recursion formula \(a_{n+1} = \frac{a_n (a_n^2 + 3)}{3a_n^2 + 1}\), \(n \in \mathbb{N}\). Show that

(a) \(a_{n+1} - 1 = \frac{(a_n - 1)^3}{3a_n^2 + 1}\) for all \(n \in \mathbb{N}\),

(b) for every initial value \(a_1 \in (0, 1)\), \((a_n)\) has a lower bound of 0 and an upper bound of 1, and is monotonically increasing,

(c) for an initial value \(a_1 \in (1, 2]\), \((a_n)\) has a lower bound of 1 and an upper bound of 2, and is monotonically decreasing.

Problem 22:  Let the recursive sequence \((a_k)\) be defined by \(a_k = a_{k-1} + 2a_{k-2}\) for \(k \geq 3\) and \(a_1 = a_2 = 1\).

(a) Using mathematical induction, show that the sequence can be also represented by \(a_k = \frac{p^k - (-1)^k}{q}\) for a suitable choice of \(p, q \in \mathbb{R}\)

(b) Determine the limit \(\lim_{n \to \infty} \sqrt[a_n]{n}\), in case it exists.

Problem 23:  Given the sequence \((a_n)\), \(a_n = \frac{n + 1}{(n+2)^2}\), show that \(a = \lim_{n \to \infty} a_n = 0\), by specifying an index \(N_0\) depending on a given \(\varepsilon > 0\), such that the following is true:

\[ |a_n - a| < \varepsilon \quad \text{for all} \quad n \geq N_0. \]

Give such an \(N_0\) explicitly for i) \(\varepsilon = \frac{1}{10}\), ii) \(\varepsilon = \frac{1}{100}\), iii) \(\varepsilon = 10^{-10}\).

Problem 24:  Compute the following limits:

(a) \(\lim_{n \to \infty} \left( \frac{n}{n+2} - \frac{n+2}{n} \right) \cdot \frac{n^2 + 3}{n+2} = \lim_{n \to \infty} \sqrt[17n^6 + 83n^4]{1} - 1\)

(b) \(\lim_{n \to \infty} \prod_{k=1}^{n} \left( 1 + \frac{1}{k} \right) - \sqrt{n^2 + 6n} = \lim_{n \to \infty} \sqrt{34\pi + 118\pi} \cdot \left[ \frac{\pi}{n^3} - 2n + 1 \right] - 1\)

Problem 25:  Determine the limits of the sequences:

(a) \(a_n = 2 + \frac{3}{4i} + \left( \frac{1}{2} + \frac{1}{3} \right)^n\) \hspace{1cm} (b) \(b_n = \frac{(3i n + 1)(2n + i)}{\sum_{k=1}^{n} ik}\)

Due date: Tuesday, December 6, 2005, 8:00 am (in the tutorial)
Problem T17: Consider the sequence

\[ a_n = \frac{1}{2} + (-1)^n \left( 1 - \frac{1}{n} \right) \]

Is the sequence bounded? If so, give the smallest possible value for \( r \), for which \( |a_n| \leq r \). Justify your answer.

Problem T18: Consider the iterative formula:

\[ a_{n+1} = \frac{1}{5}(a_n^2 + 4), \quad n \in \mathbb{N}. \]

Analyze the sequences corresponding to initial values of:

(a) \( a_1 = 2 \),  
(b) \( a_1 = 4 \)

in terms of boundedness and monotonicity. Consider in particular the quantity \( a_{n+1} - a_n \).

Aufgabe T19: Determine the limit of the sequence \((a_n)\), where

(a) \( a_n = \sqrt{n^2 + n - n} \),  
(b) \( a_n = \sqrt{4 + \frac{n - 1}{n + 1}} \),  
(c) \( a_n = \frac{n^4 - 2}{n^2 + 4} - \frac{n^3(n^2 - 3)}{n^3 + 1} \).

Aufgabe T20: Compute the limits of the sequences

(a) \( a_n = \left[ 1 + \left( -\frac{3}{5} \right)^n \right] \cdot \left[ \frac{10^n}{n!} - \frac{3n^2 + 1}{(2n + 1)^2} \right] \)
(b) \( b_n = \frac{(1 + \sqrt{5})(n + 5)}{(1 - \frac{1}{2}\sqrt{3})(2n + 3)} \),
(c) \( c_n = \sqrt{17 \cdot 2^n + 1} \left( \sqrt{n + 1} - \sqrt{n} \right) \).

Tutorial: Tuesday, November 29, 2005 (room 203 in the ID)