Worksheet 7
Advanced Mathematics I for Mechanical Engineering

Problem 31: Let

\[ a_1 = b \quad a_{k+1} = \frac{|a_k|}{2a_k - 1} \]

be a recursively defined sequence with initial values \( b = -\frac{1}{4} \), and \( b = \frac{1}{4} \).

(a) What are the possible limits of the sequence?

(b) For which of the values of \( b \) is the sequence monotone? For which is it bounded?

(c) For each value of \( b \) verify that the sequence converges and determine the limit in case it does.

Problem 32: Let \( p(x) = x^4 - 3x^3 - 3x^2 + 11x - 6 \).

(a) Determine the numbers \( a_0, \ldots, a_4 \in \mathbb{R} \), where \( p(x) = \sum_{j=0}^{4} a_j (x-1)^j \).

(b) Factorize \( p \), and determine all roots of \( p \).

Problem 33: Let the function \( f : \mathbb{R} \to \mathbb{R} \) be defined as

\[ f(x) = \begin{cases} 1 - 2x - x^2, & x \leq 1, \\ 9 - 6x + x^2, & x > 1. \end{cases} \]

Determine the largest possible intervals, in which the function is invertible. Determine the inverse function in each case and sketch it.

Problem 34: At which points \( x \in \mathbb{R} \) are the following functions \( f_j : \mathbb{R} \to \mathbb{R} \) continuous?

(a) \( f_1(x) := \begin{cases} \frac{x^3 + 4x^2 + x - 6}{x^3 - 3x + 2}, & x \in \mathbb{R} \setminus \{1, -2\} \\ 0, & x = 1, \\ -\frac{1}{3}, & x = -2, \end{cases} \)

(b) \( f_2(x) := \begin{cases} x, & x \in \mathbb{Z}, \\ 0, & \text{otherwise}. \end{cases} \)

Problem 35: The function \( f \) is defined over \( \mathbb{R} \setminus \{2\} \) by

\[ f(x) = \begin{cases} \frac{12x - 9}{x^2 - 2x}, & 1 < |x| < 3, \ x \neq 2 \\ p(x), & \text{otherwise} \end{cases} \]

where \( p \) is a polynomial. The polynomial \( p \) should be determined, so that the function \( f \) is continuous. Make an assumption for \( p \) and prove that this assumption leads to a unique solution. Determine the polynomial.

Due date: Tuesday, January 10, 2005, 8:00 am (in the tutorial)
**Problem T25:** Let 

\[ p(x) = x^4 + 8x^3 + 22x^2 + 24x + 9. \]

Expand \( p \) at \( x_0 = -2 \), i.e. find a representation in the form \( p(x) = \sum_{j=0}^{4} a_j(x+2)^j \) Additionally, factorize the polynomial.

**Problem T26:** Given are the set \( D \subset \mathbb{R} \) and the function \( f : D \to \mathbb{R} \) with the following formula:

(a) \( x \mapsto \frac{x^3 + x^2 - 4x - 4}{x^2 - x - 2} \),

(b) \( x \mapsto \frac{x^3 - 2x^2 + 3x - 2}{x^2 - 2x + 5} \).

Specify the maximal domain \( D \) of \( f \). Further, for part a), determine the range \( W \) of \( f \) and decide if an inverse function \( g : W \to D \) exists. Specify it, if it does.

**Problem T27:** Determine the limits \( \lim_{x \to x_0} f(x) \) of the following functions \( f \) and points \( x_0 \):

(a) \( f(x) = \frac{x - 2}{x^2 - 4} \) für \( x > 2 \), \( x_0 = 2 \),

(b) \( f(x) = \frac{\sqrt{x} - 1}{\sqrt{x} - 1} \) für \( x > 1 \), \( x_0 = 1 \).

**Problem T28:** Consider the piecewise defined function

\[ f(x) = \begin{cases} 12 & x < -1 \\ p(x) & -1 \leq x < 2 \\ 1 - 2x & x \geq 2, \end{cases} \]

where \( p \) is a polynomial.

(a) Determine a polynomial \( p \) with the smallest possible degree, such that \( f \) is continuous. Is it unique?

(b) Can such a polynomial be determined, so that additionally \( f(1) = -2 \)? Justify your answer in case it cannot, or specify it in case it can.

**Tutorial:** Tuesday, December 13, 2005, room 203 in the ID