Worksheet No.6
Advanced Mathematics I

Exercise 26: Examine the following sequences for boundedness, monotonicity and convergence. (Limits or accumulation points need not be specified). If appropriate, analyze suitable subsequences of each sequence.

(a) \[ a_n = \frac{1 + 6n + 2n^2}{(n+3)n}, \]
(b) \[ b_n = 6 \cdot \frac{6 + n^2}{n}, \]
(c) \[ c_n = \frac{(-2)^{-n} + 1}{1 + 2n} - 1 + \frac{2n}{1 + 2n}, \]
(d) \[ d_n = \frac{1 + 2^n}{1 + 2^n + (-2)^n}. \]

Exercise 27: Determine the limit of the sequence \( (a_n)_n \), where

(a) \[ a_n = \sqrt{n^2 + n - n}, \]
(b) \[ a_n = \sqrt[4]{4 + \frac{n-1}{n+1}}, \]
(c) \[ a_n = \frac{n^4 - 2}{n^2 + 4} + \frac{n^3(3 - n^2)}{n^3 + 1}. \]

Exercise 28: We discuss the recursively defined sequence

\[ a_1 = b, \quad a_{k+1} = \frac{|a_k|}{2a_k - 1}, \quad k \in \mathbb{N} \]

for two initial values \( b = -\frac{1}{4} \), sowie \( b = \frac{1}{4} \).

(a) Compute all possible limits under the assumption of convergence.
(b) Determine for which initial value the sequence is monotonic.
(c) Determine for which initial value the sequence is bounded.
(d) Justify whether the sequence is converging or not for either of the initial values. Determine the limits.

Exercise 29: The equation

\[ x = \frac{x^3}{4} + \frac{1}{5} \]

for \( 0 \leq x \leq 1 \) may be solved numerically by an iteration. For this, we choose the sequence

\[ x_0 = 0, \quad x_n = \frac{x_{n-1}^3}{4} + \frac{1}{5}, \quad n \in \mathbb{N}. \]

Prove convergence of \( (x_n)_n \) and give an estimate for the error after \( k \) steps.
Hint: Determine \( C \in \mathbb{R} \) such that \[ |x_{n+1} - x_n| \leq C|x_n - x_{n-1}|. \]

Exercise 30: Given a constant \( c \in \mathbb{R} \) we define the sequence \( (z_n)_n \) by

\[ z_0 = 0, \quad z_{n+1} = \frac{n^2 + 2 + c}{n+1}, \quad n \in \mathbb{N}. \]

(a) Let \( c = -2 \), give a guess for a closed representation for \( (z_n)_n \) for \( n \geq 2 \), prove this form, analyze for convergence and compute the limit.
(b) Show that the sequence is monotonously increasing for \( c = 1 \) and prove divergence by comparism with the sequence \( v_n = (4/3)^{n-2} \).

Due date: Please hand in your homework on Friday, December 14, 11:15.
Exercise T21: Analyze the following sequences to determine whether they are bounded, monotonic, and convergent (in case of convergence you need not specify the limit).

\[ a_n = \frac{1 + n + n^2}{n(n + 1)} \quad \text{,} \quad b_n = \frac{1 + n + n^2}{n + 1} \quad \text{,} \quad c_n = \frac{1}{1 + (-2)^n} \quad \text{,} \quad d_n = \frac{1 + (-2)^n}{1 + 2^n} \quad . \]

Exercise T22: Compute the limits:

(a) \( \lim_{n \to \infty} \left( \frac{n}{n + 1} - \frac{n + 1}{n} \right) \cdot \frac{n^2 + 3}{n + 2} \quad , \)

(b) \( \lim_{n \to \infty} \left( \frac{p_n}{n} \right)^n \quad \text{where} \quad p_n = \prod_{k=1}^{n} \left( 1 + \frac{1}{k} \right) \quad , \)

(c) \( \lim_{n \to \infty} \sqrt[3]{17n^6 + 83n^4} \quad . \)

Exercise T23: The sequence \( (a_n) \) is defined recursively by the initial value \( a_0 \) and the iteration formula

\[ a_{n+1} = \frac{1}{2 - a_n} \quad , \quad n > 0 . \]

(a) Show that the sequence is well defined and convergent for \( a_0 < 1 \) and compute its limit.

(b) What is the difference, when \( a_0 = 1 \), \( a_0 = \frac{5}{4} \), and \( a_0 = \frac{100}{99} \)?

Exercise T24: Which of the following assertions is true?

(a) A sequence converges, if it is monotonic and bounded.

(b) If a sequence converges, it is monotonic and bounded.

(c) If a sequence is not bounded, it can’t be convergent.

(d) If a sequence is not monotonic, it can’t be convergent.

(e) If a sequence has exactly one accumulation point, it converges.

(f) If a sequence converges, it has exactly one accumulation point.

Tutorial date: Tuesday, December 11, 2007.