Worksheet No.3
Advanced Mathematics I

Exercise 11: Compute the limits of the sequences

(a) \( a_n = \frac{2008(1 + n + n^2)}{n(n + 2009)} \),

(b) \( b_n = \sqrt{n^2 + an + b} - n \), \( a, b \in \mathbb{R} \), \( n \) sufficiently big,

(c) \( c_n = \left[ 1 + \left( -\frac{3}{5} \right)^n \right]\cdot \left( \frac{10^n}{n!} - \frac{3n^2 + 1}{(2n + 1)^2} \right) \),

(d) \( d_n = \sqrt[3]{17} \cdot 2^n \left( \sqrt{n + 1} - \sqrt{n} \right) \).

Exercise 12: Investigate if the following sequences converge. Compute the limit in case it exists.

(a) \( a_n = \left( \frac{n}{n + 1} - \frac{n + 1}{n} \right) \cdot \frac{n^2 + 3}{2n + 2} \), \( n \in \mathbb{N} \),

(b) \( b_n = \left( \frac{2p_n}{n} \right)^n \) für \( p_n = \prod_{k=1}^{n} \left( 1 + \frac{1}{k} \right) \), \( n \in \mathbb{N} \).

Hint for (b): Prove the equation \( p_n = n + 1 \) by the mathematical induction, at first.

Exercise 13: Determine the limits of the complex sequences:

(a) \( a_n = 2 + \frac{3}{4i \cdot n} + \left( \frac{1 + \frac{1}{3 \cdot i}}{2} \right)^n \),

(b) \( b_n = \frac{(3i \cdot n + 1)(2n + i)}{\sum_{k=1}^{n} ik} \).

Exercise 14: Given the sequence \( (a_n)_n \), \( a_n = \frac{n + 1}{(n + 2)^2} \), show that \( a := \lim_{n \to \infty} a_n = 0 \), by specifying an index \( N_0 \) depending on a given \( \varepsilon > 0 \), such that the following is true:

\[ |a_n - a| < \varepsilon \quad \text{for all} \quad n \geq N_0. \]

Give such an \( N_0 \) explicitly for i) \( \varepsilon = \frac{1}{10} \), ii) \( \varepsilon = \frac{1}{100} \), iii) \( \varepsilon = 10^{-10} \).

Exercise 15: Let the recursive sequence \( (a_k)_k \) be defined by \( a_k = a_{k-1} + 2a_{k-2} \) for \( k \geq 3 \) and \( a_1 = a_2 = 1 \).

(a) Using mathematical induction, show that the sequence can be also represented by \( a_k = \frac{p^k - (-1)^k}{q} \) for a suitable choice of \( p, q \in \mathbb{R} \).

(b) Determine the limit \( \lim_{k \to \infty} \sqrt[k]{a_k} \), in case it exists.

Due date: Please hand in your homework on Friday, November 21, 11:15.
Exercise T9: Determine the limit of the sequence \((a_n)_n\), where
\[ (a) \quad a_n = \sqrt{n^2 + n} - n, \quad (b) \quad a_n = \frac{n^4 - 2}{n^2 + 4} + \frac{n^3(3 - n^2)}{n^3 + 1}. \]

Exercise T10: Compute the limits of the following complex sequences:
\[(a) \quad a_n = \frac{2 + i^n}{4 + n}, \quad n \in \mathbb{N}, \quad (b) \quad b_n = \frac{(5n + i)(ni + 3)}{(1 + i)^n}, \quad n \in \mathbb{N}, \quad (c) \quad c_n = \frac{2(-6)^ni^n + 2^n}{5(-6)^{n+1}i^{n+3} + 3^n}, \quad n \in \mathbb{N}. \]

Exercise T11: Using the definition of convergence, compute the limit \(a\) of the sequence \(\left( \frac{n}{\sqrt{n^2 + 1}} \right)_n\).

Exercise T12: Consider the sequence \((a_n)_n\) with \(a_n = \frac{n-1}{n+1}, \quad n \in \mathbb{N}\). Determine an index \(N\) such that \(|a_n - 1| < \varepsilon\) for every \(n \geq N\), when
\[(a) \quad \varepsilon = \frac{1}{10}, \quad (b) \quad \varepsilon = \frac{1}{1000}, \quad (c) \quad \varepsilon > 0 \text{ is arbitrary.} \]
\[(d) \text{ Does the sequence } (a_n)_n \text{ converge? If so, to what limit?} \]

Exercise T13: Consider the sequence
\[ a_n = \frac{1}{2} + (-1)^n \left( 1 - \frac{1}{n} \right), \quad n \in \mathbb{N}. \]
Is the sequence bounded? If so, give the smallest possible value for \(r\), for which \(|a_n| \leq r\). Justify your answer.

Tutorial date: Tuesday, November 18, 2008.