Worksheet No.5
Advanced Mathematics I

Exercise 21: The sequence \((a_n)_n\) is recursively defined by

\[ a_{n+1} = a_n + \frac{\sin(a_n)}{a_n}, \quad a_0 = \frac{1}{2}. \]

(a) Show that \(0 < a_n < \pi\). Hint: You may use without any proof the inequality \(\sin(x) < \pi x - x^2\) for \(x \in (0, \pi)\).

(b) Prove that the sequence \((a_n)_n\) converges and determine the limit.

Exercise 22: Let \(f\) be defined by

\[ f(x) = \frac{x^3 - 3x + 2}{x^3 - 7x + 6}. \]

Determine the maximal domain and the range of \(f\).

Exercise 23: Given are the functions \(f\) and \(g\) with suitable real domains by

(i) \(f(x) = x^2 - 14x + 45\),
(ii) \(g(x) = \frac{1}{\sqrt{2 - x}} + 5\).

(a) Determine the maximal domain of these functions.

(b) Investigate if the functions \(f\) and \(g\) are monotonic and specify their range.

(c) Determine the sets \(D_f\) and \(D_g\) respectively, such that \(f\) is invertible on \(D_f\) and \(g\) is invertible on \(D_g\). Calculate the inverse functions, their domains and the ranges.

Exercise 24: Consider the polynomial \(p(x) = x^3 - 12x + 17\) of degree 3 in \(x \in \mathbb{R}\).

(a) Show that \(p(x)\) is strictly monotonic for \(x > 2\) by expanding it in term of powers of \(x - 2\).

(b) Show that \(p(x)\) has an inverse function \(f\) for \(x > 2\), and determine the domain and the range of \(f\). Sketch the graph of \(f\).

Exercise 25: Let \(p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0\) be a polynomial with real coefficients \(a_k, \quad k = 0, \ldots, n-1\) and \(z \in \mathbb{C}\). Prove that:

(a) If \(p(z) = 0\) then \(p(\overline{z}) = 0\).

(b) The product of all zeros of the polynomial \(p\) is real.

(c) The sum of all zeros of \(p\) is real.

Due date: Please hand in your homework on Friday, December 5, 9:45.
Exercise T18: Given are the set $D \subset \mathbb{R}$ and the function $f : D \to \mathbb{R}$ with the following formula:

$\begin{align*}
(a) \quad x & \mapsto \frac{x^3 + x^2 - 4x - 4}{x^2 - x - 2}, \\
(b) \quad x & \mapsto \frac{x^3 - 2x^2 + 3x - 2}{x^2 - 2x + 5}.
\end{align*}$

Specify the maximal domain $D$ of $f$. Further, for part (a), determine the range $W$ of $f$ and decide if an inverse function $g : W \to D$ exists. Specify it, if it does.

Exercise T19: Prove that there is no inverse function of $f : x \mapsto (x - 1/3)^2$ on $\mathbb{R}$. Show that $f : \mathbb{Z} \to f(\mathbb{Z})$ is invertible.

Exercise T20: Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 
1 - 2x - x^2, & x \leq 1, \\
9 - 6x + x^2, & x > 1.
\end{cases}$$

Determine the largest possible intervals in which the function is invertible. Determine the inverse function in each case and sketch it.

Exercise T21: Let

$$p(x) = x^4 + 8x^3 + 22x^2 + 24x + 9, \quad x \in \mathbb{R}.$$  

(a) Expand $p$ at $x_0 = -2$, i.e. find a representation in the form $p(x) = \sum_{j=0}^{4} a_j(x + 2)^j$.

(b) Factorize the polynomial $p$. 

Tutorial date: Tuesday, December 2, 2008.