Worksheet No.10
Advanced Mathematics I

Exercise 46: Deduce the derivatives of the following functions (for $a \in \mathbb{R}$):

(a) $f(x) = \cos(x)$  
(b) $f(x) = \arccos(x)$  
(c) $f(x) = a^x$  
(d) $f(x) = \log_a x$

Exercise 47: Compute the given limits. Why is it not allowed to use de l'Hôpital’s rule in part (c)?

(a) $\lim_{x \to 0} \frac{\tan x - x}{x^3}$  
(b) $\lim_{x \to \infty} \frac{x^2e^x}{(e^x - 1)^2}$  
(c) $\lim_{x \to \infty} \frac{x - \sin x}{x + \sin x}$  
(d) $\lim_{x \to \infty} (\sqrt{x^2 + 1} - \sqrt{x^2})$

Hint: Use the Mean Value Theorem for the function $x \mapsto \sqrt{x}$ in part (d).

Exercise 48: Let

$$f(x) = \frac{1}{2} \ln \left( \frac{x + 1}{x - 1} \right), \quad x > 1.$$  

(a) Compute the derivative of $f$.

(b) $f$ can also be represented by the following power series:

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{(2k + 1)x^{2k+1}}.$$  

Use this representation to compute the derivative of $f$.

Exercise 49: Show that for $x \in [-1, 1]$

$$-1 + x - \frac{x^2 - 1}{x^2 + 1} \leq \arctan(x) \leq 1 + x + \frac{x^2 - 1}{x^2 + 1}.$$  

Hint: Analyze the extrema of the terms’ differences.

Exercise 50: Between Karlsruhe and Bruchsal the suburban train S3 runs with a maximum speed of 160 km/h. The power consumption of such a train is proportional to the square of its velocity. Powering the train at a speed of 50 km/h costs 100 EUR per hour. On top of that there are fixed costs of 400 EUR per hour for the operation of the train (labour costs, maintenance etc.).

(a) At which speed are the operation costs per kilometer minimal?

(b) The mean daily revenue is 14 EUR per kilometer. How fast should the train go to maximize the total profit (i.e. the difference between revenue and costs) per hour?

Due date: Please hand in your homework on Friday, January 23, 9:45.
Exercise T34: Given are the following power series:

\[ f(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}, \quad g(x) = \sum_{k=0}^{\infty} (k+1)x^k. \]

Determine the domains, in which the derivatives of the functions can be determined by differentiating the corresponding power series, and compute these derivatives.

Exercise T35: Compute the limits:

(a) \( \lim_{x \to 0} \frac{\cos x + 3x - 1}{2x} \)
(b) \( \lim_{x \to \frac{\pi}{2}} \frac{\ln(\frac{\pi}{2} - x)}{\tan x} \)
(c) \( \lim_{x \to \infty} \frac{\sqrt{x^2 + 4}}{\sqrt{x^2 + 9}} \)
(d) \( \lim_{x \to 0} x^{\tan x} \)

Exercise T36: Use the Mean Value Theorem to prove the following inequalities:

(a) \( |\sin x - \sin y| \leq |x - y| \) for \( x, y \in \mathbb{R} \)
(b) \( |\cos e^x - \cos e^y| \leq |x - y| \) for \( x, y \leq 0 \)

Exercise T37: Show the following inequalities for \( x \in [-1, 1] \):

(a) \( 6x^3 + 3x^2 > 4x - 1 \)
(b) \( \arcsin(x) \leq \frac{x}{2} + 2x\sqrt{1 - x^2} \).

Hint: Analyze the extrema of the terms’ differences.

Tutorial date: Tuesday, January 20, 2009.