Worksheet No.12
Advanced Mathematics I

Exercise 56: Calculate the following integrals using suitable substitutions:

(a) \[ \int_0^{\frac{\pi}{2}} \cos(x) \cdot e^{\sin(x)} \, dx, \]
(b) \[ \int \frac{2x + 7}{x^2 + 7x + 3} \, dx, \]
(c) \[ \int \frac{\cos(\ln(x))}{x} \, dx, \]
(d) \[ \int \frac{1 + \ln(x)}{x - x \ln(x)} \, dx. \]

Exercise 57: Evaluate the following integrals using integration by parts:

(a) \[ \int_0^1 t \cdot \arctan(t) \, dt, \]
(b) \[ \int x^n \ln x \, dx, \quad n \neq -1, \]
(c) \[ \int \sin^2(x) \, dx. \]

Exercise 58: Determine the indefinite integral

\[ \int \frac{5e^{3x} + e^{2x} + 3e^x + 1}{e^{4x} + e^x} \, dx, \]
using a suitable substitution and a subsequent decomposition into partial fractions.

Exercise 59: For \( n \in \mathbb{N}_{>0} \) we define

\[ \Gamma(n) := \lim_{a \to \infty} \int_0^a x^{n-1} e^{-x} \, dx. \]

Show that \( \Gamma(n + 1) = n \cdot \Gamma(n) \) and use this to prove \( \Gamma(n + 1) = n! \).

Remark: This integral exists more generally for all \( n \in \mathbb{R} \setminus \{0, -1, -2, \ldots\} \) and defines a continuous function on this domain: the so-called Gamma function.

Exercise 60:
A 30 cm long shaft of brass has as cross-section a circle with diameter 4 mm, which is flattened at one side by 1 mm (see sketch). Brass has the density 8.4 g/cm\(^3\). How heavy is the shaft?

Due date: Please hand in your homework on Friday, February 6, 9:45.
Exercise T42: Compute the following integrals using suitable substitutions:

(a) $\int \frac{(1 + x)^2}{\sqrt{x}} \, dx$, (b) $\int \frac{(2 + t^2) t^3}{(1 + t^2)^3} \, dt$, (c) $\int_0^a a^r \, dr$ (for $a > 1$), (d) $\int_1^4 \frac{(1 + \sqrt{w})^3}{\sqrt{w}} \, dw$.

Exercise T43: Compute the following integrals using partial integration:

(a) $\int_0^\pi x \cdot \sin(x) \, dx$, (b) $\int (x + 2)e^{2x} \, dx$, (c) $\int x^2 \cdot \ln(x) \, dx$, (d) $\int \frac{x}{\cos^2(x)} \, dx$.

Exercise T44: Compute the following integrals using partial fraction decomposition:

(a) $\int \frac{5x^2 - 11x + 5}{x^3 - 4x^2 + 5x - 2} \, dx$, (b) $\int \frac{x^3 + 6x^2 + 3x + 18}{x^3 + x^2 + 4x + 4} \, dx$.

Exercise T45: Calculate the following integrals:

(a) $\int x^3 \sqrt{7 - 6x^2} \, dx$, (b) $\int \frac{dx}{\sqrt{x + \sqrt{x}}}$, (c) $\int e^{3x} \cosh x \, dx$.

Tutorial date: Tuesday, February 3, 2009.