Worksheet No.4
Advanced Mathematics II

Exercise 16: Compute an anti derivative of
\[ f(t) = \frac{2(tanh(t))^4 + 3(tanh(t))^3 + (tanh(t))^2 - 2}{cos(t)(tanh(t))^3 + cos(t)(tanh(t))^2 - sin(t) - cos(t)}. \]

Hint: Start with a suitable substitution – the tan(t/2)-substitution is rather complicated here.

Exercise 17: Let the function \( g : (0, \infty) \rightarrow \mathbb{R} \) be continuously differentiable, strictly monotonously increasing and onto. Determine the general solution of the differential equation
\[ u'(x) = -\frac{x^2}{x^2 - 1} \frac{g(u(x))}{g'(u(x))}, \quad x > 0. \]

Formulate this differential equation explicitely for the case \( g(x) = ln(x) \) and determine the solution which satisfies the initial value condition \( u(0) = e \). Finally, considering again \( g(x) = ln(x) \), compute all values \( a \in \mathbb{R} \) such that the initial value problem with initial condition \( u(1) = a \) has a solution.

Exercise 18: Compute the general solution of the differential equation
\[ y^{(4)}(x) - 7y'''(x) + 18y''(x) - 20y'(x) + 8y(x) = 0. \]

Exercise 19: Determine the real general solution of the differential equation
\[ u'''(x) + 3u''(x) + 9u'(x) - 13u(x) = 0, \quad x \in \mathbb{R}. \]

Show by means of the general solution that every initial value problem \( u(0) = a, u'(0) = b, u''(0) = c \) has a unique solution.

Exercise 20: Consider the inhomogeneous linear differential equations
\[ \begin{align*}
(a) \quad y''(x) - 2y'(x) - 3y(x) &= x^2 + \frac{4}{3} x + \frac{4}{3} \\
(b) \quad y'''(x) - By'(x) + y(x) &= \sin(x), \quad B \in \mathbb{R}.
\end{align*} \]

Find a polynomial of degree two which solves the differential equation in (a), and a trigonometric polynomial \( p(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) \) which solves the equation in (b). Finally, compute the constant \( B \) appearing in (b) such that \( f(x) = \sin(x) \) solves the differential equation in (b).

Due date: Please hand in your homework on Wednesday, May 14, 09:30.
Aufgabe T9: (a) Determine the general solution of the differential equation:

\[ y'''(x) + 3y''(x) + 4y'(x) + 2y(x) = 0 \]

using the ansatz \( y(x) = e^{\lambda x}, \lambda \in \mathbb{C}. \)

(b) Afterwards, determine numbers \( a, b, c \in \mathbb{R} \) such that the polynomial \( p(x) = ax^2 + bx + c \) solves the corresponding inhomogeneous differential equation for right hand side \( x^2 + 11. \)

Aufgabe T10: Determine the general real-valued solution of the homogeneous linear ordinary differential equation of third order

\[ y'''(x) + 2y''(x) + 2y'(x) + y(x) = 0 \]

using the exponential ansatz \( y(x) = e^{\lambda x}, \lambda \in \mathbb{C}. \)

Aufgabe T11: Determine for each of the following differential equations a real fundamental system:

(a) \( y^{(6)}(x) + y^{(4)}(x) - 5y^{(2)}(x) + 3y(x) = 0 \)
(b) \( y^{(8)}(x) - y^{(7)}(x) + 2y^{(6)}(x) - 2y^{(5)}(x) + y^{(4)}(x) - y^{(3)}(x) = 0. \)

Aufgabe T12: Determine a real-valued fundamental system for the homogeneous fourth-order ordinary differential equation

\[ y^{(4)}(x) - 3y'''(x) + 4y'(x) = 0. \]

Tutorial date: Wednesday, Mai 7.