Exercise 21: Determine for each of the following differential equations a real fundamental system:

(a) \(9u'(x) + 11xu''(x) + 4x^2u'''(x) + x^3u''''(x) = 0\)
(b) \(-8u(x) + 8xu'(x) + 28x^2u''(x) + 11x^3u'''(x) + x^4u''''(x) = 0\)

Exercise 22: Determine the general solution of the following differential equation
\[
2x^2z''(x) + 4x^2[\sqrt{z'(x)}]^2 + 6xz'(x) + 5 = 0, \quad x > 0.
\]
Hint: Use the substitution \(y(x) = e^{2z(x)}\).

Exercise 23:

Consider the differential equation
\[
2x^3u'''(x) + Bx^2u''(x) + xu'(x) - 10u(x) = 0, \quad x > 0.
\]

(a) Determine \(B\) so that \(u_1(x) = x^2\) is a solution of the differential equation.
(b) With the constant \(B\) from part (a) determine the general solution of the differential equation.

Exercise 24: Consider the homogeneous linear differential equation with non-constant coefficients
\[
(1 + x^2)u''(x) - 2xu'(x) + 2u(x) = 0, \quad x \in (0, \infty).
\]

(a) Verify that \(u(x) = x\) is a solution of the differential equation.
(b) Determine by reduction of order a solution \(u_2(x)\) that is linearly independent of \(u_1(x)\).
(c) The real-valued general solution is \(u(x) = C_1u_1(x) + C_2u_2(x)\) where \(C_1, C_2 \in \mathbb{R}\). Show that each initial value problem \(u(1) = a, u'(1) = b\) with \(a, b \in \mathbb{R}\) has a unique solution.

Exercise 25: Determine the general solution of the nonlinear differential equation
\[
1 + [y'(x)]^2 = 2y(x)y''(x).
\]
Hint: Consider \(x\) as function of \(y\). Define the function \(p\) with \(p(y) = y'(x(y))\) and compute \(p'(y)\). In this way one can reduce the order of the differential equation.

Due date: Please hand in your homework on Wednesday, May 28, 09:30.
### Tutorial 4
#### Advanced Mathematics II

**Aufgabe T13:** Determine the real-valued general solution of the homogeneous third-order ordinary differential equation

\[ u'''(x) - \frac{2}{x} u''(x) + \frac{5}{x^2} u'(x) - \frac{5}{x^3} u(x) = 0, \quad x > 0. \]

**Aufgabe T14:** Find the real-valued general solution for each of the following homogeneous ordinary differential equations:

(a) \( x^2 u''(x) - 5x u'(x) + 13u(x) = 0, \quad x > 0, \)
(b) \( u''(x) - 5u'(x) + 13u(x) = 0, \quad x > 0, \)
(c) \( u'''(x) - \frac{3}{x} u''(x) + \frac{7}{x^2} u'(x) - \frac{8}{x^3} u(x) = 0, \quad x > 0. \)

**Aufgabe T15:** Consider the differential equation

\[(1 - x)y'''(x) + xy'(x) - y(x) = 0.\]

(a) Check that \( y(x) = x \) is a solution.
(b) Determine the solution of the following initial value problem

\[ y(0) = 1, \quad y'(0) = 3. \]

Hint: Via reduction of order determine first a solution \( y_2(x) \) that is linearly independent of \( y_1(x) \).

**Aufgabe T16:** Determine the solution of the following initial value problem

\[ y''(x) = 2y(x)y'(x), \quad y(0) = 1, \quad y'(0) = 2. \]

Hint: Consider \( x \) as function of \( y \). Define the function \( p \) with \( p(y) = y'(x(y)) \) and compute \( p'(y) \). In this way one can reduce the order of the differential equation.

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**Tutorial date:** Wednesday, May 14.