Worksheet No. 10
Advanced Mathematics II

Exercise 46: Determine the Laplace-transform of:

(a) \( f(x) = x^2 + 3x + 4 + x^2 \sin(2x) \)
(b) \( f(x) = \begin{cases} \sin(x) & 0 \leq x < \pi \\ \cos(x) & x \geq \pi \end{cases} \)
(c) \( f(x) = (e^{2x} + e^{3x}) \cdot \sin(4x) \)
(d) \( f(x) = \cos(x) - \sin(x) = (x \cdot \cos(x))' \)
(e) \( f(x) = x^n, \ n \in \mathbb{N} \)

Use in part (e) the definition of the Laplace-transform and apply mathematical induction.

Exercise 47: Specify which functions \( f(t), t \in [0, \infty) \) have been Laplace-transformed in the given examples:

(a) \( F(s) = \frac{2s}{s^4 + 2s^3 + 2s^2 + 2s + 1} \)
(b) \( F(s) = \arccot(s - 1) \)
(c) \( F(s) = \frac{e^{-\pi s}}{\sqrt{s^2 + 1}} \)

Hint: Apply in (a) the partial fraction decomposition. In parts (b) and (c) adopt appropriate computational rules of the Laplace-transform.

Exercise 48: Solve the initial value problem

\[ y'''(x) + 4y''(x) + 5y'(x) + 2y(x) = x, \quad y(0) = 0, \ y'(0) = 1, \ y''(0) = -1 \]
by means of a Laplace-transform.

Exercise 49: Determine the solution of the initial value problem

\[ y'(x) = \begin{pmatrix} 4 & 4 \\ -2 & 0 \end{pmatrix} y(x), \quad y(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \ x \geq 0, \]
by means of a Laplace-transform.

Exercise 50: Consider the vector space \( C[0,1] \) with its scalar product \( \langle \cdot, \cdot \rangle \) as defined in exercise No.10 on 2\textsuperscript{nd} worksheet. The functions \( v_j(x) = \sin(\pi j x), \ j = 1, \ldots, n \) span a subspace \( U \) of \( C[0,1] \). We have \( v_j \perp v_k \) for \( j \neq k \). Let \( u = \sum_{j=1}^{n} a_j v_j \in U \) (where \( a_j \in \mathbb{R} \)) be a function which satisfies \( \langle u'' - u, v \rangle = \langle f, v \rangle \) with \( f(x) = x \) for all \( v \in U \).

(a) Show that \( \langle u'' - u, v_k \rangle = -\frac{1}{2} a_k (\pi^2 k^2 + 1) \) for \( k = 1, \ldots, n \).
(b) Verify that \( \langle f, v_k \rangle = \frac{(-1)^{k+1}}{\pi k} \) for \( k = 1, \ldots, n \).
(c) Determine the coefficients \( a_k \) and find the approximative solution \( u \) for \( n = 3 \).

Note: For the case \( U = C[0,1] \) we obtain an exact solution of the differential equation \( u''(x) - u(x) = f(x) \). The restriction on a subset \( U \subset C[0,1] \) yields an approximative solution for the initial value problem \( u(0) = u(1) = 0 \).
In mathematics this approach is called Galerkin method. It is applied particularly for the finite element method.

Due date: Please hand in your homework on Thursday, July 2, 11:15 a.m.
Exercise T37: Determine the Laplace-transforms of the following real-valued functions:

(a) $f(t) = \cosh(t)$  (b) $f(t) = t \cosh(t)$  (c) $f(t) = \sinh(\omega t)$  (d) $f(t) = \cos^2(\omega t), \quad \omega > 0.$

Exercise T38: Specify which functions $f(t)$ have been Laplace-transformed in the given examples:

(a) $F(s) = \frac{1}{s^5}$  (b) $F(s) = \frac{6}{(s-2)^4}$  (c) $F(s) = \frac{1}{s(s+1)^2}$  (d) $F(s) = \frac{2s}{(s+1)^2(s^2+1)}.$

Exercise T39:

(a) Determine $\mathcal{L}((3-t^2) \sin t - 3t \cos t)(s)$.

(b) Solve the initial value problem

$$u^{(4)}(t) + 2u''(t) + u(t) = \sin t, \quad u(0) = u'(0) = u''(0) = u'''(0) = 0.$$ 

Exercise T40: Solve the initial value problem by means of the Laplace-transform:

\[
\begin{align*}
x'(t) &= 2y(t) + 1 \\
y'(t) &= -2x(t) + 2t \\
x(0) &= 0, \quad y(0) = 1
\end{align*}
\]