Worksheet No. 11
Advanced Mathematics II

Exercise 51: Consider the functions

\[ h(t-a) = \begin{cases} 
0, & t < a \\
1, & t \geq a 
\end{cases} \quad \text{and} \quad f(t) = \begin{cases} 
t-1, & 1 \leq t < 3, \\
8-2t, & 3 \leq t < 4, \\
0, & \text{else} 
\end{cases} \]

(a) Rewrite the function \( f \) into a form without any case differentiation using the Heaviside step function \( h(t-a) \).

(b) Sketch the graph of the function \( f \) and determine its Laplace-transform.

Exercise 52: Check if the given parameter integral

\[ J(t) = \int_{0}^{1} \arcsin(tx) \, dx, \quad 0 \leq t < 1, \]

is differentiable. Evaluate it by first determining its derivative \( J'(t) \) on the open interval \( 0 < t < 1 \). From this result, derive \( J(t) \), \( 0 \leq t < 1 \), and determine the value of the integration constant from the value of \( J(t) \) at \( t = 0 \). Can \( J(t) \) be smoothly continued at \( t = 1 \)?

Exercise 53: Evaluate the convolution \( f \ast g \) for the functions given by

(a) \( f(t) = \sin(t) \), \( g(t) = \begin{cases} 
0, & 0 \leq t < 3 \\
2, & t \geq 3, 
\end{cases} \)

(b) \( f(t) = \cosh(t) \), \( g(t) = \sin^2(t) \).

Also determine \( \mathcal{L}(f \ast g) \) and compare the result with \( \mathcal{L}f \cdot \mathcal{L}g \).

Exercise 54: Given is an initial value problem

\begin{align*}
-3x'(t) + x(t) + 2y'(t) + 2y(t) &= 6e^t + 5e^{-2t}, \\
x'(t) - x(t) + y'(t) - y(t) - z'(t) + z(t) &= \frac{3}{2} + \frac{3}{2} e^{-2t}, \\
2x'(t) - y(t) - z'(t) &= 0,
\end{align*}

with conditions \( x(0) = 2, \quad y(0) = 3, \quad z(0) = 4 \).

Determine the first component of the solution, namely the function \( x : [0, \infty) \to \mathbb{R} \). Hint: Apply the Laplace-transform.

Exercise 55: A car with a wheelbase of 3 m and equal hubloads of \( F = 7200 \) N is parked on a 30 m long bridge, 6 m in from the right bearing. The bridge has the special load per unit of length of \( q_0 = 14400 \) N/m and the bending stiffness of \( EJ = 6 \cdot 10^9 \) [Nm²]. Given these characteristics, the displacement \( w \) of the bridge satisfies the ordinary differential equation

\[ EJ \cdot w'''(x) = q_0 + F\delta(x-21) + F\delta(x-24), \quad w(0) = w''(0) = w'(30) = w''(30) = 0. \]

Determine the function \( w(x) \) by means of a Laplace transformation. Hint: You may set \( w'(0) = A \) and \( w'''(0) = B \), and obtain the constants \( A \) and \( B \) at the end from the conditions \( w(30) = w''(30) = 0 \).

Due date: Please hand in your homework on Thursday, July 9, 11:15 a.m.
Exercise T41: Given the function \( \Lambda(x) = \begin{cases} 
0, & 0 \leq x < 1 \\
2 - x, & 1 \leq x < 2 \\
0, & \text{else}
\end{cases} \)

(a) determine the Laplace-transform \( \mathcal{L}\Lambda(s) \),
(b) describe \( f(x) \) by means of \( \Lambda(x) \) in closed form:
\[
f(x) = \begin{cases} 
4x, & 0 \leq x < 1 \\
-x + 5, & 1 \leq x < 2 \\
x^2 - 5x + 9, & 2 \leq x < 3 \\
-x^2 + 4x, & 3 \leq x < 4 \\
0, & x \geq 4
\end{cases}
\]
(c) use this to describe the Laplace-transform \( \mathcal{L}f(s) \) by means of \( \mathcal{L}\Lambda(s) \).

Exercise T42: Verify the continuity and differentiability of the parameter integral
\[
J(t) = \int_0^1 \arctan(tx) \, dx.
\]
Compute its derivative for \( t \in \mathbb{R} \). Does the limit of the derivative exist for \( t \to 0 \)?

Exercise T43: Determine the convolution \( f \star g \) for the following pair of functions
(a) \( f(t) = t^2; \quad g(t) = 1 - h(t - 1) \), where \( h(t) = \begin{cases} 
0, & t < 0 \\
1, & t \geq 0
\end{cases} \).
(b) \( f(t) = \sinh t; \quad g(t) = \sin 2t \).
Compute the Laplace-transform \( \mathcal{L}(f \star g) \) and compare the result with \( \mathcal{L}f \cdot \mathcal{L}g \).

Exercise T44: Solve the initial value problem
\[
-x'(t) + \frac{3}{2}y'(t) - 5x(t) + 2y(t) + z(t) = 0, \\
x(0) = 1, \quad y(0) = -1, \quad z(0) = -3, \quad z'(0) = 2
\]
by means of Laplace-transform.

Tutorial date: Monday, July 6, 3:45 p.m.–5:15 p.m.