Exercise 6: Let 
\[ u = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 0 \\ 3 \\ -1 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad z = \begin{pmatrix} 2 \\ 2 \\ 3 \\ -3 \end{pmatrix} \]

be vectors in \( \mathbb{R}^4 \).

(a) Compute the following linear combinations: \( u + v - z \), \( 2v - (w - z) \), \( 2u - v + 2w \).

(b) Show that the vectors \( u, v, w \) are a basis of the subspace span \{u, v, w\}.

(c) What is the dimension of the subspace span \{u, v, w, z\}?

Exercise 7: Determine all linear combinations of

(a) \( a^{(1)} = (4, 1, 1)^\top \), \( a^{(2)} = (1, 2, 3)^\top \), \( a^{(3)} = (5, 6, 7)^\top \),

(b) \( b^{(1)} = (2, 1, 1)^\top \), \( b^{(2)} = (1, -1, 6)^\top \), \( b^{(3)} = (5, 1, 8)^\top \),

(c) \( c^{(1)} = (5, 1, 4)^\top \), \( c^{(2)} = (4, 1, 3)^\top \), \( c^{(3)} = (-1, 3, -4)^\top \)

which describe \( x = (3, 1, 2)^\top \).

Exercise 8: Consider the plane \( E : 4x_1 + x_3 + 8 = 0 \), the point \( P = (2|1|1) \) and the line \( H : x(\lambda) = (4, 3, -2)^\top + \lambda(3, 1, -1)^\top \), \( \lambda \in \mathbb{R} \).

(a) Determine a line \( G \) through \( P \) that is orthogonal to \( E \).

(b) Determine the distance from \( P \) to \( E \) as well as the point \( Q \) in \( E \) closest to \( P \).

(c) Determine the point at which the line \( H \) intersects \( E \) and the point \( R \) on \( H \), that is closest to \( P \).

Exercise 9: Let the points \( P = (2|1|0), Q = (1|3| - 1) \) and \( R = (0|2|0) \) be given.

(a) Represent the plane \( E \) through the points \( P, Q \) and \( R \) in both parametric and normal form.

(b) Does the line \( G : x(u) = (-2, -7, 0)^\top + u(3, 2, 1)^\top \) intersect the plane \( E \)? If so determine the point and angle of intersection.

(c) Compute the orthogonal projection of the direction vector \( (3, 2, 1)^\top \) of the line \( G \) onto the normal vector of the plane \( E \). Using this and the intersection point of \( G \) and \( E \) determine the projection \( H \) of the line \( G \) onto \( E \).

Exercise 10: \( C[0,1] \) denotes the vector space of continuous functions on the closed interval \([0, 1]\). Let \( U \) be a subspace of \( C[0,1] \) spanned by two polynomials \( b^{(1)}(x) = 1 \) and \( b^{(2)}(x) = x - \frac{1}{2} \). We define \( y(x) := \sqrt{x} \in C[0,1] \) and the scalar product of two functions by

\[ \langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} \, dx \in \mathbb{C}. \]

(a) Find a linear combination \( c = a_1 b^{(1)} + a_2 b^{(2)} \in U \), such that \( c(0) = y(0) \) and \( c(1) = y(1) \).

(b) Determine \( d \in U \) with smallest distance to \( y \), i.e. the distance vector \( e = d - y \) must be orthogonal to \( b^{(1)} \) and \( b^{(2)} \). Draw the graphs of \( y \) and the approximations \( c \) and \( d \) of it in \([0, 1]\) in a figure.

Remark: The Finite Element Method includes the computing of orthogonal approximations like \( d \).

Due date: Please hand in your homework on Thursday, May 7, 11:15.
Exercise T5: Let
\[ u = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}, \quad v = \begin{pmatrix} -9 \\ 2 \\ 4 \end{pmatrix}, \quad w = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} \]
be vectors in \( \mathbb{R}^3 \).

(a) Compute the following linear combinations of these vectors: \( u + w \), \( v - 3u \), \( 2u - v + w \).

(b) Show that every pair of vectors in the set \( \{u, v, w\} \) are linearly independent.

(c) Are the three vectors also linearly independent as a triple?

Exercise T6: For which \( \alpha \in \mathbb{R} \) is \( x = (-7, \alpha, 2)^T \) a linear combination of \( a^{(1)} = (1, 2, 4)^T \), \( a^{(2)} = (-2, 1, 2)^T \) and \( a^{(3)} = (3, 1, 2)^T \)? Determine all possible linear combinations of \( x \).

Exercise T7: Let
\[ E : x_1 - x_3 = 0 \quad \text{and} \quad F : x_1 + 2x_2 + x_3 = 4. \]
be planes in \( \mathbb{R}^3 \).

(a) Find the intersection line \( G \) between \( E \) and \( F \).

(b) For another straight line \( H \), which lies neither in \( E \) nor in \( F \), exist only the following possibilities:

- it intersects \( E \) as well as \( F \) in only one point. (Find the angles of intersection.)
- it intersects one of them in only one point, but doesn’t intersect the other at all,
- it doesn’t intersect any of them.

Construct an example for each of these possibilities and visualize the geometric position of the planes and the straight line.

Exercise T8: Consider the points \( P = (2|1| - 4), Q = (-1| - 5| - 1) \) and the plane \( E : x_1 + 2x_2 - x_3 = 2. \)

(a) Determine a parametric representation of the line \( G \) through \( P \) and \( Q \), as well as a parametric representation of \( E \).

(b) Compute the point \( S \) of intersection of \( G \) and \( E \), and show that the line \( G \) and the plane \( E \) intersect at right angles (i.e. orthogonally). How far is \( P \) from \( E \)?

Tutorial date: Monday, May 4, 3:45pm-5:15pm