Worksheet No. 6
Advanced Mathematics II

Exercise 26: Let the following differential equation for \( x > 0 \) be given:
\[
y'''(x) - \frac{2}{x}y''(x) + \frac{5}{x^2}y'(x) - \frac{5}{x^3}y(x) = 0
\]
Check, if the following functions are solutions of this differential equation:
(a) \( y_1(x) = \sin(x^2) \),
(b) \( y_2(x) = x \),
(c) \( y_3(x) = \exp\left(\frac{2}{x}\right) \),
(d) \( y_4(x) = x^2 \cos(\ln(x)) \).

Exercise 27: Determine the general real-valued solution of the homogeneous differential equation
\[
y'''(x) + 2y''(x) + 2y'(x) + y(x) = 0
\]
using the exponential ansatz \( y(x) = e^{\lambda x}, \lambda \in \mathbb{C} \).

Exercise 28: Solve the initial value problems:
(a) \( y''(x) - 2y'(x) - 3y(x) = 0, \ x \in \mathbb{R}, \ y(0) = 0, \ y'(0) = -4 \);
(b) \( y'''(x) - 4y'(x) = 0, \ x \in \mathbb{R}, \ y(0) = 0, \ y'(0) = 0, \ y''(0) = 2 \).

Exercise 29: Determine the real-valued general solution of the differential equation
\[
x^4u'''(x) + 6x^3u''(x) - 2ux'(x) + 20u(x) = 0, \ x > 0.
\]

Exercise 30: Show that \( u(x) = e^{x^2} \) solves the homogeneous differential equation
\[
u''(x) - 2xu'(x) - 2u(x) = 0, \ x \in (0, \infty).
\]
Determine a second non-trivial solution by means of the method of reduction of the order.

Due date: Please hand in your homework on Thursday, June 4, 11:15 a.m.
Exercise T21: Determine the general solution of the linear homogeneous differential equations with constant coefficients:

(a) $y'''(x) - 3y''(x) - y'(x) + 3y(x) = 0$, $x \in \mathbb{R}$,

(b) $y'''(x) + 7y''(x) + 19y'(x) + 13y(x) = 0$, $x \in \mathbb{R}$.

Exercise T22: Consider the homogeneous Euler differential equation

$$2x^3u'''(x) + Bx^2u''(x) + xu'(x) - 10u(x) = 0, \quad x > 0.$$ 

(a) Determine $B \in \mathbb{R}$ so that $u_1(x) = x^\frac{5}{2}$ is a solution of the differential equation.

(b) Determine the general solution of the differential equation for the computed constant $B$ from part (a).

Exercise T23: Determine the real general solution of the differential equation

$$u'''(x) + 3u''(x) + 9u'(x) - 13u(x) = 0, \quad x \in \mathbb{R}.$$ 

Show by means of the general solution that every initial value problem $u(0) = a$, $u'(0) = b$, $u''(0) = c$ has a unique solution.

Exercise T24: Show that $y(x) = x$ fulfils the differential equation

$$(1 + x^2)y''(x) - 2xy'(x) + 2y(x) = 0, \quad x \in \mathbb{R}$$

and determine other solution by means of the method of reduction of the order.

Tutorial date: Monday, June 1, 3:45 p.m.–5:15 p.m.