Exercise: When determining the angle dependency of solutions of oscillation problems in spherical coordinates (for example, electromagnetic fields, acoustics, electron orbitals) one comes across the Legendre differential equations

\[(1-x^2)f''(x) - 2x f'(x) + n(n+1)f(x) = 0, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}, x > 1.\]

(a) Show that for \(n = 1\) the function \(f(x) = x\) fulfills this equation.
(b) Determine the general solution for the case \(n = 1\).

Solution. (a) \(f(x) = x \Rightarrow f'(x) = 1, \quad f''(x) = 0\)

Insert it into the DE:

\[(1-x^2)0 - 2x \cdot 1 + 1(1+1)x = 0 \quad \checkmark \quad \text{Thus} \quad f(x) = x \quad \text{fulfills the DE for} \quad n = 1.
Solution (b) This DE is homogeneous, linear, and of the 2nd order. The solution set is a 2-dim. vector space! From part (a) we know that 
\[ f(x) = e^x \cdot x \], satisfies the given DE, where \( c \in \mathbb{R} \).

We need a second solution, which is lin. indep. to \( x \), to find!

Method: reduction of the order. Make the ansatz: \( f(x) = C(x) \cdot x \).

Then \( f'(x) = C'(x) \cdot x + C(x) \cdot 1 \), \( f''(x) = C''(x) \cdot x + C'(x) + C(x) = C''(x) \cdot x + 2C'(x) \).

Insert the ansatz into the DE:
\[
(1-x^2) \left( C''(x) \cdot x + 2C'(x) \right) - 2x \left( C'(x) \cdot x + C(x) \right) + 2C(x) = 0
\]
\[
- f''(x) = f'(x).
\]

Set \( df(x) = C'(x) \):

\[
(1-x^2) \left( \frac{d}{dx} x \cdot \frac{dx}{dx} \right) = 2x^2 \frac{d}{dx} C(x) - (1-x^2) \cdot 2 \cdot C(x)
\]

Solving method? Remember AMI. \( \Rightarrow \) separable DE! (separation of variables).

\[
\frac{d}{dx} \left( C(x) \cdot x \right) = \frac{2x^2}{1-x^2}, \quad \int \frac{dx}{x-x^2} = \int \frac{-2}{x^2} + \frac{1}{x} + \frac{1}{1+x} \, dx
\]

\[
\ln \left| C(x) \cdot x \right| = -2 \left( \ln x - \ln |x+1| + \ln |x-1| \right) + C = \frac{ln x^2 (x^2 - 1)}{x^2 - 1}.
\]

Set to zero, because this is enough to find only one solution!

(\text{take whole equation as } \text{Power of } e)

\[
e^{C(x)} = \frac{1}{x^2 (x-1)}. \quad \text{Choose } \frac{d}{dx} C(x) = \frac{-1}{x^2 (x-1)}. \quad \text{(We are looking only one solution!)}
\]

Thus \( C(x) = \int \frac{dx}{x^2 (x-1)} = \int \left( \frac{1}{x^2} + \frac{1}{x-1} + \frac{1}{x+1} \right) \, dx
\]

\[
= x^{-1} + \frac{1}{2} \ln |x-1| - \frac{1}{2} \ln |x+1| = x^{-1} + \frac{1}{2} \ln \frac{x-1}{x+1}.
\]

Insert \( C(x) \) into the ansatz:

\[
f(x) = C(x) \cdot x = x + \frac{x}{2} \ln \frac{x-1}{x+1}. \quad \text{lin. indep. to } x.
\]

So finally we can give the general solution of the given equation:

\[
y(x) = C_1 \cdot x + C_2 \cdot \left( 1 + \frac{x}{2} \ln \frac{x-1}{x+1} \right), \quad c_1, c_2 \in \mathbb{C}.
\]

\( \text{accords to a plane through the origin in parameter form!} \)
3rd Exercise: Let a weight of mass \(m\) be placed on a horizontal line and assume it is fixed by a horizontal spring with spring constant \(k\). The amplitude, i.e., the distance between the weight and the quiescent point, at the time \(t\) we denote as \(x(t)\). Set up the corresponding differential equation for \(x(t)\) and determine the general (real) solution.

\[
F = -D \dot{x}(t)
\]

**Solution:**

\[
x''(t) = a(t) = F(t) - \frac{1}{m} = \frac{1}{m}(-D) \cdot x(t) = -\frac{D}{m} x(t).
\]

So we get the DE:

\[x''(t) + \frac{D}{m} x(t) = 0.\]

of 2nd order, i.e., the solution set is a 2D vector space.

Type of the DE: hom., linear, with constant coefficients.

Ansatz: \(x(t) = e^{\lambda t}, \lambda \in \mathbb{C} \Rightarrow x(t) = \lambda e^{\lambda t}, \dot{x}(t) = \lambda^2 e^{\lambda t}\).

**Inserting into the Ansatz into the equation** gives: we obtain:

\[
\lambda^2 e^{\lambda t} + \frac{D}{m} e^{\lambda t} = 0
\]

\[
\Rightarrow e^{\lambda t} (\lambda^2 + \frac{D}{m}) = 0 \Rightarrow \lambda^2 + \frac{D}{m} = 0 \Rightarrow \lambda^2 = -\frac{D}{m} \Rightarrow \lambda = \pm \sqrt{-\frac{D}{m}} = \pm \sqrt{i} \lambda_0.
\]

**The general solution** is:

\[
x(t) = c_1 e^{\lambda_0 t} + c_2 e^{-\lambda_0 t}, \text{ e.g. } c_1, c_2 \in \mathbb{C}.
\]

**Complex function, but we are looking for a real-valued function.**

**Idea:** use the formula: \(e^{i\theta} = \cos \theta + i \sin \theta\):

\[
x(t) = c_1 \cos \left( \sqrt{-\frac{D}{m}} t \right) + i c_1 \sin \left( \sqrt{-\frac{D}{m}} t \right) + c_2 \cos \left( -\sqrt{-\frac{D}{m}} t \right) + i c_2 \sin \left( -\sqrt{-\frac{D}{m}} t \right) = c_1 e^{i\sqrt{-\frac{D}{m}} t} + c_2 e^{-i\sqrt{-\frac{D}{m}} t} = c_1 e^{\sqrt{-\frac{D}{m}} t} + c_2 e^{-\sqrt{-\frac{D}{m}} t}, c_1, c_2 \in \mathbb{C}.
\]

**Free parameters!**

Here we see the dimensions of the solution set.

**Note:** \(x(t)\) is real-valued if and only if \(c_1^* = c_1\) and \(c_2^* = c_2\).

**Alternative way:** \(e^{i\alpha}, e^{-i\alpha} \rightarrow e^{i\alpha} + e^{-i\alpha} = 2 \cos \alpha, e^{i\alpha} - e^{-i\alpha} = 2i \sin \alpha\).

\[
(\text{Alternative way}: e^{i\alpha}, e^{-i\alpha} \rightarrow e^{i\alpha} + e^{-i\alpha} = 2 \cos \alpha, e^{i\alpha} - e^{-i\alpha} = 2is \sin \alpha).
\]