Exercise 16: Which of the products $AB, AX, BX, X^T A, (A^T X)^T B, B^T X, XX^T$ involving the matrices given below are well defined? Where appropriate determine the size of the resulting matrix and then evaluate the product.

$$A = \begin{pmatrix} 3 & 7 \\ 2 & 8 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 8 \\ 6 & 5 \end{pmatrix}, \quad X = \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}.$$

Exercise 17: Given the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},\quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix},\quad C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 3 & 4 \end{pmatrix},$$

calculate the inverses of $A$ and $B$ and find a matrix $D \in \mathbb{R}^{3 \times 3}$ such that $ABD = C$.

Hint: The matrix multiplication is not commutative, i.e. $AB \neq BA$ in general.

Exercise 18: Given the 3 vectors $b_1 = (1,0,1)^T, b_2 = (-3,2,1)^T, b_3 = (0,-4,1)^T$ in $\mathbb{R}^3$, determine the matrix $A$ of the linear map $\Phi$ with $\Phi(e_1) = b_1, \Phi(e_2) = b_2, \Phi(e_3) = b_3$, where $e_j$ denotes the $j$th coordinate unit vector.

Also given the 3 vectors $c_1 = (-2,1,2)^T, c_2 = (3,0,-4)^T, c_3 = (-5,0,7)^T$. show that the matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 7 & 4 & 5 \\ 4 & 2 & 3 \end{pmatrix}$$

defines a linear map $\Psi : x \mapsto Bx$, with $Bc_j = e_j, j = 1,2,3$. Also show that the linear map $\Lambda : x \mapsto (AB)x$ satisfies $\Lambda c_j = b_j, j = 1,2,3$.

Exercise 19: Given be two matrices representing a reflection and a rotation, respectively, namely

$$S = \frac{1}{3} \begin{pmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{pmatrix} \quad \text{and} \quad R = \frac{1}{2} \begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ -\sqrt{3} & 0 & 1 \end{pmatrix}.$$

(a) Show that $S$ and $R$ are orthogonal matrices.

(b) Determine the plane of reflection associated with $S$.

(c) Determine the axis of rotation and the angle of rotation associated with $R$. Hint: $\arccos(1/2) = \frac{\pi}{3}$.

Exercise 20: Given the matrices

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & -4 \\ 3 & 6 & 4 \end{pmatrix},$$

(a) show $A = L \cdot R$.

(b) Now set $b = (3,6,-11)^T$. Solve the linear system $Ax = b$, by calculating a vector $y$ which satisfies $Ly = b$, and then a vector $x$ with $Rx = y$.

Due date: Please hand in your homework until Friday, 14 May, 12:00 into the AM2-box near seminar room Z1, building 01.85 (Fritz-Erler-Str. 1–3).
**Exercise T10:** Consider the three matrices

\[
A = \begin{pmatrix} 1 & 5 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & -1 \\ 0 & 3 \\ 2 & 1 \end{pmatrix}.
\]

Decide which of the following products are defined and compute them if possible:

\[
AB, BA^\top, CA, CA^\top, C^\top A^\top, B^\top C^\top A^\top, (BA^\top)^\top C^\top, (CB)^\top A.
\]

**Exercise T11:** Figure 1 shows a square around the origin with the vertices \(A, B, C, D\). The following figures show the images of the square obtained by applying different linear mappings \(\Phi_{1,2,3} : \mathbb{R}^2 \to \mathbb{R}^2\):

Determine the matrices corresponding to the linear mappings. Describe in each case the inverse map (if it exists). Calculate the inverse matrices (if possible).

**Exercise T12:**

(a) Find all fixed points of the matrix

\[
A = \begin{pmatrix} -2 & 2 & 1 \\ 2 & -4 & 3 \\ 1 & 3 & -3 \end{pmatrix},
\]

(i.e. all \(x \in \mathbb{R}^3\) which satisfy \(Ax = x\)) using Gaussian elimination.

(b) The fixed points of \(A\) all lie on a straight line \(g\). Find a vector \(w\) orthogonal to \(g\). Calculate the angle between \(Aw\) and \(g\).

(c) Let \(w\) be an arbitrary vector orthogonal to \(g\). Show that then also \(Aw\) is orthogonal to \(g\).

(d) Give a reason why \(x \mapsto Ax\) cannot be a rotation around \(g\).

**Tutorial date:** Friday 7 May