Worksheet No.5
Advanced Mathematics II

Exercise 21: Given $A = \begin{pmatrix} 1 & 4 & 3 & 0 \\ 4 & 7 & 6 & 0 \\ 7 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 9 \\ 11 \\ 0 \end{pmatrix}$,

(a) compute all solutions $x_b$ of the homogeneous linear system $Ax = 0$. These solutions form a subspace of $\mathbb{R}^4$. Determine a basis of this subspace.

(b) Show: $x_p = (2, 1, -1, -2)^T$ is a solution of the inhomogeneous linear system $Ax = b$. Determine the set of all solutions of $Ax = b$ by using part (a).

Exercise 22: Compute the following determinants:

(a) $D = \begin{vmatrix} 3 & \frac{1}{3} & 2 & \pi \\ 1 & a & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 3 & 1 \end{vmatrix}$,

(b) $D = \begin{vmatrix} 2 & 4 & 2 & -1 \\ 2 & 3 & 0 & 5 \\ 2 & 1 & 2 & 3 \\ 1 & 2 & 0 & 2 \end{vmatrix}$,

(c) $D = \begin{vmatrix} 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ -1 & 2 & 0 & 3 \\ 6 & 2 & 3 & 1 \end{vmatrix}$.

Exercise 23: Compute the determinant of the matrix $C := (AB)^{-1}$ with

$A := \begin{pmatrix} 1 & 2 & 3 & -2 & 2 \\ 3 & -1 & 4 & 2 & 0 \\ 8 & -3 & -2 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $B := \begin{pmatrix} -1 & 3 & -2 & 2 & -1 \\ 0 & 2 & -3 & -3 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 5 \\ 0 & 2 & 0 & 0 & 6 \end{pmatrix}$.

Exercise 24: Let

$A = \begin{pmatrix} 1 & \alpha & -2 \\ 2 & -1 & \alpha - 1 \\ -1 & \alpha + 1 & 3 \end{pmatrix} \in \mathbb{R}^{3 \times 3}$.

Calculate the determinant of $A$. For which $\alpha \in \mathbb{R}$ is $A$ invertible? Calculate $A^{-1}$ for $\alpha = -1$.

Exercise 25: The linear map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is given by the matrix

$A = \frac{1}{2} \begin{pmatrix} 7 & 10 \\ -5 & -7 \end{pmatrix}$

with respect to the standard basis $E = \{e^{(1)}, e^{(2)}\}$. Another basis $B = \{b^{(1)}, b^{(2)}\}$ of the vector space $\mathbb{R}^2$ is given by $b^{(1)} = (-1, 1)^T$ and $b^{(2)} = (-3, 2)^T$.

(a) Determine the basis transformation matrix from $B$ to $E$, i.e. a matrix which transforms the vectors $x = \alpha_1 b^{(1)} + \alpha_2 b^{(2)} = (\alpha_1, \alpha_2)^T_B$ with respect to the basis $B$ into the representation $x = \beta_1 e^{(1)} + \beta_2 e^{(2)} = (\beta_1, \beta_2)^T_E$.

(b) Determine the basis transformation matrix from $E$ to $B$, i.e. a matrix which transforms the vectors with respect to the standard basis $E$ into the vectors with respect to the basis $B$.

(c) Determine the matrix of the map $\Phi$ with respect to the basis $B$. Describe $\Phi$ geometrically.

Due date: Please hand in your homework until Thursday, 20 May, 12:00 into the AM2-box near seminar room Z1, building 01.85 (Fritz-Erler-Str. 1–3).
Exercise T13: Calculate the determinant of a real-valued $4 \times 4$ matrix

\[
\begin{vmatrix}
1 & 2 & \pi & 1 \\
2 & 4 & 1 & 0 \\
1 & 0 & 0 & 3 \\
2 & 4 & -1 & 0
\end{vmatrix}
\]

(a) using the expansion rule across the 3rd row,
(b) using the expansion rule across the 4th column,
(c) via Gaussian elimination.

Exercise T14: Given

\[
A = \begin{pmatrix}
5 & i - 1 & 7 & -4 \\
5 & \frac{1}{2}(1 - i) & 5 & -3 \\
4 & 0 & 4 & -2 \\
0 & 0 & 1 & -1
\end{pmatrix} \in \mathbb{C}^{4 \times 4} \quad \text{and} \quad B = \begin{pmatrix}
1 & -i - 1 & -2 & -2 \\
2 & -i - 1 & -4 & -4 \\
3 & -5i & -5 & -4 \\
4 & -7i - 7 & -6 & -7
\end{pmatrix} \in \mathbb{C}^{4 \times 4}.
\]

Determine $\det(A)$ and $\det(B)$, as well as $\det(AB^*)$ and $\det(A^{-1}B)$.

Exercise T15: Compute the determinant of the matrix

\[
A = \begin{pmatrix}
3 & \alpha - 1 & -\alpha + 3 \\
0 & \alpha + 3 & -4 \\
0 & 2 & \alpha - 3
\end{pmatrix} \in \mathbb{R}^{3 \times 3}.
\]

For which $\alpha$ is the linear system of equations $A\alpha x = b$ for $b = (9, 4, -2)^T$ solvable?

Tutorial date: Friday 14 May