Exercise 36: Solve the initial value problem

$$u'(t) = A \cdot u(t) \quad t \in [0, \infty) \quad u(0) = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix}.$$

Exercise 37: Determine the general solution of the following differential equations for $x > 0$

(a) $y''(x) - y(x) = x$ using the method of undetermined coefficients,

(b) $y''(x) - y(x) = \frac{1}{x}$ using the method of variation of parameters.

Hint: The integral $\int \frac{e^x}{x} dx$ has no simple closed-form. You may leave this unevaluated in your result.

Exercise 38: Solve the initial value problem

$$y'''(x) + 3y''(x) + 4y'(x) - 8y(x) = (7 - 13x)e^x, \quad x \in \mathbb{R}$$

with $y(0) = y''(0) = 2$ and $y'(0) = 0$.

Exercise 39: Determine the general solution of the following differential equation

$$x^2y''(x) + xy'(x) - 4y(x) = 1 + x^2, \quad x > 0.$$

Exercise 40: Two pressure tanks with different capacities $V_1$ and $V_2$ are linked by a pipe which is closed by a valve. Before opening the stop valve at time $t = 0$ the air in the tanks has two different pressures $p_1(0)$ and $p_2(0)$.

By the ideal gas law $pV = nRT$ ($n$ amount of substance, $R$ gas constant, $T$ temperature) assuming isothermal balancing we achieve the relation $\dot{p}V = \dot{n}RT$ and finally with flow resistance $W$ of the pipe, $\dot{n} = W\dot{p}$ and the notation $a_{1,2} := \frac{RT}{WV_{1,2}}$ the following system for the model

$$\begin{pmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \end{pmatrix} = \begin{pmatrix} -a_1 & a_1 \\ a_2 & -a_2 \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}, \quad t > 0.$$

Let $p_1(0) = 1$ bar, $p_2(0) = 9$ bar, $a_1 = 1$ bar/s and $a_2 = 3$ bar/s.

(a) In which tank and when is the pressure equal to two bar?

(b) Which pressure will be obtained when the system is completely balanced?
Tutorial No.8
Advanced Mathematics II

Exercise T22:
(a) Determine a real fundamental system for
\[
\begin{align*}
    u' &= 2u + 2v \\
v' &= -\frac{1}{2}u + 2v.
\end{align*}
\]
(b) Solve the following initial value problem
\[
u'(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} u(x), \ x \in \mathbb{R}, \ u(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]

Exercise T23: Consider the inhomogeneous linear second-order ordinary differential equation
\[-15u(x) + 3xu'(x) + x^2u''(x) = 8x^{-3}, \ x > 0.\]
(a) Find a real-valued fundamental system of the associated homogeneous differential equation.
(b) Find a particular solution by the method of variation of parameters. Determine the general solution of the inhomogeneous problem.

Exercise T24: Determine the characteristic polynomial for the inhomogeneous third-order ordinary differential equation
\[y'''(x) + y''(x) + 4y'(x) + 4y(x) = f(x), \ x \in \mathbb{R}.\]
For each of the following right hand sides determine an ansatz which reflects the structure of the right-hand side (method of undetermined coefficients):
\[
\begin{align*}
f_1(x) &= (4 + 12x)e^{2x} \\
f_2(x) &= e^{-x} \\
f_3(x) &= \sin(2x) \\
f_4(x) &= x^2 \cos(2x) \\
f_5(x) &= xe^{-x} \sin(2x) \\
f_6(x) &= (2 + x) \sin(2x)
\end{align*}
\]
Use this to find a particular solution for \(f_1\) and \(f_2\).

Tutorial date: Friday 11 June