Exercise sheet 4
Advanced Mathematics III

**Question 16:** Evaluate the domain integrals

\[ (a) \int_B (x_1^2 - x_2^2) \, d(x_1, x_2), \quad \text{and} \quad (b) \int_S \frac{\sin(x_2)}{x_2} \, d(x_1, x_2); \]

the first on \( B \subseteq \mathbb{R}^2 \) which is bounded by the graphs of the functions \( x_2 = x_1^2 \) and \( x_2 = x_3^3 \), the second on \( S \subseteq \mathbb{R}^2 \) which is defined by \( S := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2 \leq \frac{3}{2}\} \).

Which variant proves to be more effective: integrating first w.r.t. \( x_1 \) or integrating first w.r.t. \( x_2 \)?

**Question 17:** To be evaluated is the domain integral

\[ J = \int_0^2 \left[ \int_{x_2=0}^{x_1} \frac{x_1}{x_2+5} \, dx_2 \right] dx_1 + \int_1^2 \left[ \int_{x_2=0}^{\sqrt{20-x_1^2}} \frac{x_1}{x_2+5} \, dx_2 \right] dx_1. \]

Sketch the domain of integration in the \((x_1, x_2)\)-plane and give an explicit representation of its boundary curves. Change the order of integrations to compute the value of \( J \). (Remark: Determining \( J \) in the given order of integration will not be accepted as a solution.)

**Question 18:** For each case of the following domain integrals, sketch the domains of integration \( D \) and introduce coordinates adapted conveniently to the geometries of \( D \) in order to facilitate evaluation of the integrals:

(a) \( D \) be the triangle with vertices \((0, 0)\), \((1, 0)\) and \((1/2, 1/2)\). Evaluate

\[ \int_D e^{x_1 + x_2} \, dx. \]

(b) \( D \) be that part of an annulus centred on \((0, 0)\), with outer radius 4 and inner radius 2, which lies in the half plane \( x_2 < 0 \). Evaluate

\[ \int_D (x_1^2 - x_2^2) \, dx. \]

(c) \( D \) be that part of an ellipse with semi–minor axis 1 w.r.t. \( x_1 \) and semi–major axis 2 w.r.t. \( x_2 \), which is cut out by the lines \( x_2 = x_1 \) and \( x_2 = -x_1 \), with \( x_1 \geq 0 \). Evaluate

\[ \int_D (x_1 - x_2) \, dx. \]

**Question 19:** Consider the surface \( F \subseteq \mathbb{R}^3 \) given by \( F = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 4 + x_1 x_2 \} \). Let \( K \) be that body, which is bounded by (i) the cylinder \( x_1^2 + x_2^2 = 4 \), (ii) the “bottom” \( x_3 = 0 \), and (iii) the “lid” \( F \).

(a) Sketch the contour lines of \( F \) for \( x_3 = 0, x_3 = 1, x_3 = 2, x_3 = 3, x_3 = 4 \).

(b) Determine the volume of \( K \).

**Question 20:** The domain \( B \subseteq \mathbb{R}^2 \) is bounded by the curves \( x_2 = x_1 - 1, x_2 = \frac{1}{2}x_1 - 1, x_2 = 2x_1^3 - 1 \) and \( x_2 = \sqrt{x_1} - 1 \). Use the coordinate transformation \( x = \Psi(y) = (\frac{y_1}{y_2}, \frac{1-y_2}{y_2})^T \) to evaluate the domain integral

\[ \int_B \frac{dx}{(x_2 + 1)^3}. \]

**Deadline:** Thursday, November 20, 2008 at 15:45h