Exercise sheet 9
Advanced Mathematics III

Question 41:
(a) For each of the following partial differential equations of first order over \( \mathbb{R}^2 \), state whether they are linear, quasi-linear, or neither.

(i) \( 3x_2 \frac{\partial u(x)}{\partial x_1} + x_1^2 \frac{\partial u(x)}{\partial x_2} + x_1 u(x) = x_1 x_2 \)

(ii) \( u(x) \frac{\partial u(x)}{\partial x_1} - \left( \frac{\partial u(x)}{\partial x_2} \right)^2 + u(x) = 0 \)

(iii) \( 2x_2 u(x) \frac{\partial u(x)}{\partial x_1} + x_1 \frac{\partial u(x)}{\partial x_2} - u(x)^2 = e^{x_1 + x_2} \)

(b) Consider the following three partial differential equations of second order:

(i) \( 2 \frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} + 3 \frac{\partial^2 u(x)}{\partial x_2^2} + 2 \frac{\partial u(x)}{\partial x_1} - 5 \frac{\partial u(x)}{\partial x_2} + \sin(x_1 + x_2) u(x) = \cos(x_1 - x_2), \quad x \in \mathbb{R}^2 \)

(ii) \( x_3 \frac{\partial u(x)}{\partial x_3} + \frac{\partial^2 u(x)}{\partial x_1^2} + 2 \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} + x_1 x_2 \frac{\partial^2 u(x)}{\partial x_2^2} - 2x_1 \frac{\partial u(x)}{\partial x_1} + \ln |x_1 + x_2| = 0, \quad x \in \mathbb{R}^3 \)

(iii) \( x_2^2 \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} - 4u(x) + \frac{\partial u(x)}{\partial x_1} - x_2^2 \frac{\partial^2 u(x)}{\partial x_2^2} = 0, \quad x \in \mathbb{R}^2 \)

For each differential equation, determine its principal part and state for which points in \( \mathbb{R}^2 \) (resp. \( \mathbb{R}^3 \)) the differential equation is of the elliptical, parabolical or hyperbolical kind.

Question 42: Determine a parametric representation for the characteristics of the partial differential equation of first order

\[ x_1 x_2 \frac{\partial u(x)}{\partial x_1} - x_2^2 \frac{\partial u(x)}{\partial x_2} = x_1, \quad x_1, x_2 > 0. \]

Under all solution surfaces \( z = u(x_1, x_2) \), find the one which contains the curve given by \( x_1 = 1 \) and \( 2x_2 z = 3 \).

Question 43: Use the method of characteristics to solve the initial value problem

\[ x_1 \frac{\partial u(x)}{\partial x_1} + x_2 \frac{\partial u(x)}{\partial x_2} + (x_1^2 + x_2^2) u(x) = 0, \quad u(x_1, -x_1^2) = \exp \left( -\frac{1}{2} x_1^2 \right), \quad x_1 > 0, x_2 \in \mathbb{R}. \]

Question 44: Determine the solution \( u : \mathbb{R}^2 \to \mathbb{R} \) of the linear initial value problem

\[ 2 \frac{\partial u(x)}{\partial x_1} + 3x_2^2 \frac{\partial u(x)}{\partial x_2} + 4x_1 u(x) = 0, \quad u(0, x_2) = x_2 - 5, \quad x \in \mathbb{R}^2. \]

Question 45:
(a) Determine a number \( a \in \mathbb{R} \) such that the function \( u(x, t) = \frac{1}{4} e^{-|x|^2} \) solves the diffusion equation

\[ a \frac{\partial u}{\partial t} - \Delta u = 0 \]

for \( x \in \mathbb{R}^2 \) and \( t > 0 \).

(b) Let \( d \in \mathbb{R}^3 \setminus \{0\} \). For which vectors \( p \in \mathbb{R}^3 \) is the vector field \( E : \mathbb{R}^3 \to \mathbb{R}^3 \) with \( E(x) = p e^{i d \cdot x} \) a solution of the time harmonic Maxwell’s equations (with \( H : \mathbb{R}^3 \to \mathbb{R}^3 \))

\[ \text{rot } E - i |d| H = 0 \quad \text{and} \quad \text{rot } H + i |d| E = 0? \]

Deadline: Thursday, January 8, 2009 at 15:45h