Indicator Functions for Shape Reconstruction Related to the Linear Sampling Method

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Abstract

We provide exact shape reconstruction formulas in the spirit of the Linear Sampling method for a class of inverse problems in shape determination in the context of time-independent partial differential equations. To this end, we prove a general theorem how, and under which assumptions, domain characterizations based on the range of the square root of an operator transform into domain characterizations based on the operator itself. To show the flexibility of this general theory we then apply this general principle to a variety of shape determination problems in inverse acoustic and electromagnetic scattering theory and inverse elliptic boundary value problems.

1 Introduction

We consider inverse shape determination problems for elliptic partial differential equations. Examples include for instance inverse scattering problems where one seeks to find the shape of a scatterer from measured far field data of acoustic or electromagnetic waves, or inverse elliptic boundary value problems, where the most prominent application is electrical impedance tomography.

To tackle these problems we apply a version of the Linear Sampling method first introduced in [7, 9] for shape identification problems in scattering theory. Let us briefly recall that the method is based on the far field operator \( F \), since it tries to determine the shape of the scatterer via approximate solutions to the far field equation \((Fg)(\hat{x}) = \exp(-ik \hat{x} \cdot z)\), \( \hat{x} \in S^{d-1} = \{ x \in \mathbb{R}^d, |d| = 1 \} \), for sampling points \( z \in \mathbb{R}^d \) from a grid covering a domain of interest. Since the far field operator is compact, a regularization scheme must be applied to this linear problem: The typical choice is Tikhonov regularization,

\[
g_\varepsilon^z = (\varepsilon I + F^* F)^{-1} F^* e[z] \quad \text{where } e[z](\hat{x}) := \exp(-ik \hat{x} \cdot z),
\]

combined with a discrepancy principle to choose the value of the regularization parameter \( \varepsilon > 0 \). The shape of the scatterer is then found as the set of those points \( z \) where the norm of \( g_\varepsilon^z \) is above a certain cut-off value that enters the algorithm as a parameter. Note that the theory on this method remains somewhat incomplete as the theoretical backbone of the method is a theorem claiming that for points inside the scattering object there exists an approximate

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solution to the far field equation that blows up as the regularization parameter tends to zero. However, there is no guarantee that the Tikhonov regularization \( g^\varepsilon \) from (1) behaves in the same way. However, an alternative indicator function was already proposed in [7].

\[
z \mapsto \left| \int_{S^{d-1}} \exp(ik \cdot \theta) g^\varepsilon(z) dS(\theta) \right|, \quad z \in \mathbb{R}^d.
\]  

In [2] we showed that this variant of the Linear Sampling method yields indeed a mathematically sound way to characterize a Dirichlet scattering object in \( \mathbb{R}^3 \) since, roughly speaking, the technique is equivalent to the Factorization method, first developed in [13].

For other scattering problems, such results have not yet been published, despite they certainly are of interest in, e.g., electromagnetic inverse scattering problems (see, e.g., the discussion in the end of Chapter 3.3 of [5]).

In this paper, we prove that a generalization of (2) applied to a general inverse shape identification problem for a time-independent partial differential equation analogously yields a mathematically rigorous way of determining the shape of an obstacle. The main assumption of this generalization is that, roughly speaking, the Factorization method can be applied to the same shape identification problem. We give detailed examples for suitable shape identification problem in obstacle and electromagnetic medium scattering as well as in impedance tomography, and further note without going into details several problems the technique can also be applied (e.g., low-frequency electromagnetics or -statics or time-independent Stokes flows).

Going beyond the scope of [2], we also show that the alternative formulation of the Linear Sampling method does not only work when Tikhonov regularization is used. Indeed, arbitrary linear regularization schemes defined using regularizing filters can be employed to achieve the same theoretical properties for the method.

Additionally, we provide a regularization theory for the presented alternative formulation of the Linear Sampling method that is able to cope with noisy measurements. The shape reconstruction criterion is shown to converge point wise as the noise level tends to zero if the regularization parameter of the scheme respects several bounds that are determined, roughly speaking, by the noise level.

Let us finally note that the alternative formulation of the Linear Sampling method for scalar, acoustic scattering problems is based on the Herglotz wave functions. The analogous reformulation for electromagnetic problems is naturally based on dot products of curls of electromagnetic Herglotz wave functions with polarization vectors. The generalization to inverse elliptic problems as impedance tomography relies on dot products of gradients of layer potentials for the corresponding Green’s function with similar polarization vectors.

Let us briefly outline the structure of this paper: In the next Section 2 we present three inverse shape identification problems for which we provide the details on how to apply the general theory developed in the subsequent Section 3. In Section 4 we provide regularization theory for the alternative Linear Sampling method for general linear regularization schemes. The final Section 5 contains a couple of numerical examples for the case of penetrable dielectric media in electromagnetics.

**Notation:** We consider partial differential equations in a subset \( \Omega \) of the \( d \)-dimensional Euclidean space \( \mathbb{R}^d, \ d \geq 2 \). Considering either scalar- or vector-valued problems, we rely on real- or complex-valued functions or distributions with \( m \in \mathbb{N} \) components. These distributions typically will belong to Hilbert spaces \( H^s(U, \mathbb{R}^m) \) or \( H^s(U, \mathbb{C}^m) \) for some set \( U \subset \mathbb{R}^d \) and some regularity parameter \( s \in \mathbb{R} \). The symbols \( x, y \) and \( \hat{x} \) denote points in \( \mathbb{R}^d \) and unit
vectors in $\mathbb{S}^{d-1} = \{ y \in \mathbb{R}^d, |y| = 1 \}$, respectively. The symbol $D$ is reserved for bounded Lipschitz domains in $\mathbb{R}^d$ that play the role of scattering objects in the different settings under investigation; the exterior unit normal field to $\partial D$ is denoted by $\nu$. The ball of radius $R > 0$ around a point $x \in \mathbb{R}^d$ is $B_R(x) = \{ y \in \mathbb{R}^d, |x - y| < R \}$.

2 Applications of the Factorization Method

In this section we present three basic settings where the Factorization method can be applied to rigorously characterize penetrable or impenetrable inclusions in a background medium from the knowledge of solutions to partial differential equations that involve these inclusions. In detail, we consider acoustic and electromagnetic scattering problems involving impenetrable and penetrable scatterers, respectively, as well as an inverse shape identification problem in impedance tomography.

2.1 Scattering of Acoustic Waves from an Impenetrable Obstacle

The Helmholtz equation with wave number $k > 0$ models scattering of linear time-harmonic acoustic waves in the exterior of a bounded, impenetrable obstacle $D \subset \mathbb{R}^d$, $d = 2$ or $3$,

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^d \setminus D.$$  \hfill (3)

We assume in the following that $D$ is a bounded Lipschitz domain and distinguish three different boundary conditions on $\partial D$. In detail, we consider either a sound soft (Dirichlet), a sound-hard (Neumann), or a Robin boundary condition with real-valued coefficient $\tau \in L^\infty(\partial D, \mathbb{R})$ for the total wave field $u : \mathbb{R}^d \setminus \overline{D} \rightarrow \mathbb{C}$ and denote this boundary condition by $B(u) = 0$,

$$B(u) := u|_{\partial D} \quad \text{for a sound-soft obstacle},$$  \hfill (4)

$$B(u) := \left[ \frac{\partial u}{\partial \nu} + \tau u \right]|_{\partial D} \quad \text{for a sound-hard (}\tau = 0\text{) or a Robin (}\tau \neq 0\text{) obstacle.}$$  \hfill (5)

When an incident wave $u^i$ satisfying the Helmholtz equation in all of $\mathbb{R}^3$ illuminates the obstacle $D$ there arises a scattered field $u^s$ such that the total field can be written as $u = u^i + u^s$ in $\mathbb{R}^3 \setminus D$. Moreover, the scattered field propagates away from the obstacle, a fact that is mathematically encoded by requiring $u^s$ to satisfy Sommerfeld’s radiation condition, see, e.g., [10, Section 2.2]. If $u^s$ satisfies this radiation condition, then

$$u^s(x) = \frac{\exp(ik|x|)}{4\pi|x|^{(d-1)/2}} u^\infty(\hat{x}) + O \left( \frac{1}{|x|^2} \right) \quad \text{as} \quad |x| \rightarrow \infty, \quad \text{uniformly in} \quad \hat{x} = \frac{x}{|x|}.$$  

We call the function $u^\infty : \mathbb{S}^{d-1} \rightarrow \mathbb{C}$ the far field pattern of $u^s$. Any solution to the Helmholtz equation that either satisfies Sommerfeld’s radiation condition is called a radiating solution.

For the special setting of scattering of incident plane waves $x \mapsto \exp(ik \theta \cdot x)$ with direction $\theta \in \mathbb{S}^{d-1}$, we denote by $u^\infty(\cdot, \theta)$ the far field pattern of the scattered field corresponding to this incident plane wave. The inverse scattering problem we consider is to find the domain $D$ given the far field patterns $u^\infty(\hat{x}, \theta)$ for all $\hat{x}, \theta \in \mathbb{S}^{d-1}$.

For all the above boundary conditions, the scattered field $u^s$ is a radiating solution to

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D}, \quad B(u^s) = -B(u^i) \quad \text{on} \quad \partial D.$$  \hfill (6)
Either integral equation methods \[18\] or variational techniques involving exterior Dirichlet-to-Neumann operators \[20\] show that (6) possesses a unique radiating solution \( u^s \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus D) \) that is bounded in terms of \( B(u^s) \). More precisely, setting \( X = H^{1/2}(\partial D) \) in the sound-soft case and \( X = H^{-1/2}(\partial D) \) both in the sound-hard case and in the Robin case, it holds that for all \( R > \text{diam}(D) \) there is \( C = C(R) \) such that \( \|u^s\|_{H^1(B_R(0) \setminus D)} \leq C(R)\|B(u^s)\|_X \). Further, \( u^\infty \) is analytic in both variables and the far field operator

\[
F : L^2(S^2) \rightarrow L^2(S^2), \quad (Fg)(\hat{x}) = \int_{S^2} u^\infty(\hat{x}, \theta) g(\theta) \, dS(\theta), \quad \hat{x} \in S^{d-1},
\]

is compact. Note that linearity of the scattering problem (6) moreover implies that \( Fg \) is the far field pattern of the scattered field corresponding to an incident Herglotz wave function

\[
v_g(x) = \int_{S^2} \exp(ik x \cdot \theta) g(\theta) \, dS(\theta), \quad x \in \mathbb{R}^3.
\]

For all three boundary conditions described in \[4,5\], the far field operator is normal (see, e.g., \[15\] Th. 7.15) and possesses a complete system of eigenvectors \( v_j \in L^2(S^2) \) with corresponding eigenvalues \( \lambda_j \in \mathbb{C} \). In three dimensions all eigenvalues \( \lambda_j \) lie on the circle with center \( 8\pi i/k \) and radius \( 8\pi^2/k \); in two dimensions, they lie on the circle with center \( \exp(3\pi i/4)\sqrt{2\pi/k} \) and radius \( \sqrt{2\pi/k} \). The compactness of \( F \) implies that \( \lambda_j \to 0 \) as \( j \to \infty \).

Let us represent the eigenvalues \( \lambda_j \) as

\[
\lambda_j = |\lambda_j| \exp(i\delta) \exp(i\beta)
\]

with phases \( \delta_j \) in the interval \([0, \pi]\) and a dimension-dependent phase shift \( \beta = 0 \) in three and \( \beta = \pi/4 \) in two dimensions. For all three boundary conditions under consideration and independent of the dimension \( d \), the phases \( \delta_j \) are even contained in a sub-interval of \([0, \pi]\) of length strictly smaller than \( \pi \) (see, e.g., \[15\] for the three-dimensional case and \[8\], \[4\] Th. 7.14 as well as \[13\] Section 5) for the two-dimensional case). For the Dirichlet scattering problem it holds for instance that all \( \delta_j \in [\delta, \pi] \) for some \( \delta > 0 \), while for the Neumann and Robin problem \( \delta_j \in [0, \pi - \delta] \) for some \( \delta > 0 \).

Let us now introduce the Herglotz operator \( H : L^2(S^2) \rightarrow X \),

\[
(Hg) = B(v_g) \quad \text{on } \Gamma,
\]

that relies on the Herglotz wave function \( v_g \) from \[8\]. For all three boundary conditions \(4,5\) it is obvious that \( H \) is bounded. If \( G : X \rightarrow L^2(S^{d-1}) \) denotes the solution operator for the exterior scattering problem (6) mapping the boundary datum in \( X \) to the far field of the solution \( u^s \), then

\[
F = GH
\]

The main result of the Factorization method applied to the above-introduced inverse scattering problem for impenetrable obstacles is the following: Depending on the choice of the boundary condition, assume that \( k^2 \) is not an interior Dirichlet-, Neumann- or Robin eigenvalue. Then the function

\[
\theta \mapsto e[z](\theta) := \exp(-ik \theta \cdot z) \in L^2(S^2), \quad \text{parametrized } z \in \mathbb{R}^3,
\]

belongs to the range of \( (F^*F)^{1/4} \) if and only if the point \( z \) belongs to the obstacle \( D \). For details and proofs of this characterization we refer to Sections 1.4, 1.6 and 2.1–2.2 in \[15\].
2.2 Scattering of Electromagnetic Waves from a Non-Absorbing Medium

As a second example for an application of the Factorization method, we consider the scattering of electromagnetic waves by an inhomogeneous non-absorbing non-magnetic medium. Denote by \( \omega \) the circular frequency, by \( \varepsilon_0 \) the electric permittivity and by \( \mu_0 \) the magnetic permeability in vacuum. An electromagnetic-field propagating in \( \mathbb{R}^3 \) is a solution to the Maxwell system

\[
\text{curl} \, E - i \omega \mu_0 H = 0, \quad \text{curl} \, H + i \omega \varepsilon_0 E = 0 \quad \text{in} \, \mathbb{R}^3. \tag{11}
\]

We assume that the incident field \((E^i, H^i)\) satisfying \((11)\) is scattered by a bounded, non-conducting inhomogeneity characterized by a space-dependent permittivity \(\varepsilon\). In this situation the total field \((E, H)\) is a solution to the Maxwell system

\[
\text{curl} \, E - i \omega \mu_0 H = 0, \quad \text{curl} \, H + i \omega \varepsilon E = 0 \quad \text{in} \, \mathbb{R}^3. \tag{12}
\]

The direct scattering problem is completed by requiring that the scattered field \(E^s = E - E^i\), \(H^s = H - H^i\) satisfies the well-known Silver-Müller radiation condition at infinity, see, e.g. \([10] \text{ Section 6.2}\). As a consequence, the scattered field has the asymptotic behaviour

\[
\begin{pmatrix} E^s(x) \\ H^s(x) \end{pmatrix} = \frac{\exp(i k |x|)}{4 \pi |x|} \begin{pmatrix} E^\infty(\hat{x}) \\ H^\infty(\hat{x}) \end{pmatrix} + O \left( \frac{1}{|x|^2} \right), \quad \text{as} \, |x| \to \infty, \quad \text{uniformly in} \, \hat{x} = \frac{x}{|x|}.
\]

As in the case of an acoustic scattering problem, the far field patterns \((E^\infty, H^\infty)\) are analytic functions on \(S^2\). Moreover, we have \(E^\infty(\hat{x}) \cdot \hat{x} = H^\infty(\hat{x}) \cdot \hat{x} = 0\) and \(E^\infty(\hat{x}) = H^\infty(\hat{x}) \times \hat{x}\).

In what follows, we work with the magnetic field only. As presented in detail in \([15] \text{ Section 5.2}\), the electric field can be eliminated from the system \((12)\). We obtain the equation

\[
\text{curl} \left[ \frac{1}{\varepsilon_r} \text{curl} \, H^s \right] - k^2 H^s = \text{curl} \, f \quad \text{in} \, \mathbb{R}^3 \tag{13}
\]

with \(f = q \, \text{curl} \, H^i\), the wavenumber \(k = \omega \sqrt{\varepsilon_0 \mu_0}\) and the relative permittivity \(\varepsilon_r = \varepsilon/\varepsilon_0\). The contrast \(q\) is defined as \(q = 1 - 1/\varepsilon_r\). We make the assumption that for some bounded Lipschitz domain \(D \subseteq \mathbb{R}^3\) and some constants \(c_1, c_2 > 0\), it holds that \(\varepsilon_r \geq c_1\) and \(\varepsilon_r - 1 \geq \pm c_2\) in \(D\). Extending \(\varepsilon_r\) by 1 outside \(D\), we have \(\overline{D} = \text{supp} \, q\). Lastly we will assume that \((13)\) admits a unique radiating variational solution in \(H_{\text{loc}}(\text{curl}, \mathbb{R}^3)\) for all compactly supported \(f \in L^2(\mathbb{R}^3, \mathbb{C}^3)\).

The inverse problem which can be solved by the Factorization method is to determine \(D\) from the knowledge of the far field patterns \(H^\infty\) for all plane incident waves

\[
H^i(x) = pe^{ik \theta \cdot x}, \quad x \in \mathbb{R}^3, \tag{14}
\]

where \(p \in \mathbb{C}^3 \setminus \{0\}\) denotes the amplitude vector, \(\theta \in S^2\) the direction of incidence and we have \(p \cdot \theta = 0\). To make plain the dependence on all parameters, we will write \(H^i(x, \theta, p)\) for the incident plane wave with direction of incidence \(\theta\) and amplitude \(p\) as well as \(H^\infty(\hat{x}, \theta, p)\) for the far field of the corresponding scattered magnetic wave. Denoting by \(L^2_t(S^2)\) the space of all square-integrable tangential vector fields on the unit sphere,

\[
L^2_t(S^2) = \{ u \in L^2(S^2, \mathbb{C}^3), u(\hat{x}) \cdot \hat{x} = 0, \hat{x} \in S^2 \}\]
we introduce the far field operator $F : L^2_\ell(S^2) \to L^2_\ell(S^2)$ by

$$Fg(\hat{x}) = \int_{S^2} H^\infty(\hat{x}, \theta, g(\theta)) \, ds(\theta), \quad g \in L^2_\ell(S^2), \quad \hat{x} \in S^2. \quad (15)$$

Note that $H^\infty$ depends linearly on the polarization $p$ of the incident plane wave and thus also $F$ is a linear operator. Further properties of $F$ in this setting are given in Theorem 5.7 in [15], the most important of which for our purposes is that $F$ is compact and normal and thus possesses a complete orthonormal system of eigenfunctions. Moreover, the eigenvalues $\lambda_j$ of $F$ are all of the form $\lambda_j = |\lambda_j| e^{i\delta_j}$ with $0 \leq \delta_j < \pi$ for all $j \in \mathbb{N}$ and $\lim_{j \to \infty} \delta_j = 0$.

Introducing the Herglotz operator $H : L^2_\ell(S^2) \to L^2(D, \mathbb{C}^3)$ by

$$Hg(x) = \text{curl} \int_{S^2} g(\theta) e^{ik\theta \cdot x} \, ds(\theta), \quad x \in D,$$

we see that $Fg$ is the far field pattern of $H^*$ for the incident field $Hg$. Furthermore (see [15 Theorem 5.10]), we have the factorization

$$F = H^* TH,$$

with $T : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$ given by $Tf = q (f + \text{curl} v)$ and $v \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ is the unique radiating variational solution of

$$\text{curl} \left[ \frac{1}{\varepsilon_r} \text{curl} v \right] - k^2 v = \text{curl} [q f] \quad \text{in} \ \mathbb{R}^3.$$

Even though the definition of $H$ and $T$ slightly differ from the presentation in [15 Chapter 5], neither the operator $F$ nor the range of $H^*$ change. In the case where $k^2$ is not a transmission eigenvalue (see Definition 5.8 in [15]), we conclude that $D$ can be characterized as the set of points $z \in \mathbb{R}^3$ for which the function

$$e[z, p](\hat{x}) = ik (\hat{x} \times p) e^{-ik\hat{x} \cdot z}, \quad p \in \mathbb{C}^3 \setminus \{0\}, \quad \hat{x} \in S^2,$$

belongs to the range of $(F^* F)^{1/4}$.

We can rewrite the operator $H$ by exchanging differentiation and integration to obtain

$$Hg(x) = \int_{S^2} \text{curl} \left( g(\theta) e^{ik\theta \cdot x} \right) \, ds(\theta) = \int_{S^2} ik (\theta \times g(\theta)) e^{ik\theta \cdot x} \, ds(\theta), \quad x \in D.$$

Extending the definition of $Hg$ to all of $\mathbb{R}^3$, it holds for $z \in \mathbb{R}^3$, $p \in \mathbb{C}^3$ and $g \in L^2_\ell(S^2)$ that

$$p \cdot Hg(z) = -ik \int_{S^2} p \cdot (g(\theta) \times \theta) e^{ik\theta \cdot z} \, ds(\theta)$$

$$= -ik \int_{S^2} g(\theta) \cdot (\theta \times p) e^{ik\theta \cdot z} \, ds(\theta) = \langle g, e[z, p] \rangle_{L^2_\ell(S^2)}.$$
2.3 Electrical Impedance Tomography

Given a conductivity \( \gamma \) inside a body \( \Omega \subseteq \mathbb{R}^d \) and a current density \( g \) on the boundary \( \Gamma = \partial \Omega \), the electric potential \( u \) in \( \Omega \) is a solution to the Neumann boundary value problem

\[
\text{div}(\gamma \nabla u) = 0 \quad \text{in} \quad \Omega, \quad \gamma \frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \Gamma.
\]

(16)

We will assume throughout that the conductivity is real valued but possibly anisotropic, i.e.,

\[
\gamma \in L^\infty_+(\Omega, \mathbb{R}^{d \times d}) := \{ \sigma \in L^\infty(\Omega, \mathbb{R}^{d \times d}) \mid \sigma(x)^\top = \sigma(x) \text{ and } \theta^\top \sigma(x) \theta \geq c \text{ for some } c > 0, \text{ all } \theta \in S^{d-1} \text{ and for almost all } x \in \Omega \}.
\]

Since all conductivities will be real-valued we can restrict ourselves to real-valued function spaces. Setting

\[
L^2_\partial(\Gamma) = \{ g \in L^2(\Gamma), \int_\Gamma g \, dS = 0 \} \quad \text{and} \quad H^1_\partial(\Omega) = \{ u \in H^1(\Omega), \int_\Gamma u \, dS = 0 \},
\]

Poincaré’s inequality and Lax-Milgram’s lemma imply that the variational formulation corresponding to (16)

\[
\int_\Omega \gamma \nabla u \cdot \nabla v \, dx = \int_\Gamma gv \, dS \quad \text{for all } v \in H^1_\partial(\Omega),
\]

(17)

possesses a unique solution \( u \in H^1_\partial(\Omega) \).

In impedance tomography, one seeks to determine properties of the conductivity \( \gamma \) from boundary measurements of the electric potential on \( \Gamma \). Mathematically, the boundary measurements of a voltage potential are encoded in the so-called Neumann-to-Dirichlet operator. This operator \( \Lambda : L^2_\partial(\Gamma) \to L^2_\partial(\Gamma) \) maps \( g \in L^2_\partial(\Gamma) \) to \( u|_\Gamma \), where \( u \in H^1_\partial(\Omega) \) is the unique solution to (17).

Assume now, additionally, that \( \gamma \) differs from a known background conductivity \( \gamma^0 \in L^\infty_+(\Omega, \mathbb{R}^{d \times d}) \) by a perturbation \( Q \) defined in a Lipschitz domain \( D \) such that \( D \subset \Omega \) and such that the support \( \overline{D} \) has a connected complement in \( \mathbb{R}^d \),

\[
\gamma = \begin{cases} 
\gamma^0 + Q & \text{in } D, \\
\gamma^0 & \text{in } \Omega \setminus \overline{D},
\end{cases}
\]

where \( Q \in L^\infty(\Omega, \mathbb{R}^{d \times d}) \) is again real-valued and symmetric positive definite. Denote by \( \Lambda_0 \) the corresponding Neumann-to-Dirichlet operator mapping \( g \) to \( u_0|_\Gamma \), where \( u_0 \in H^1_\partial(\Omega) \) is the variational solution to (17) with \( \gamma \) replaced by \( \gamma^0 \). Both \( \Lambda \) and \( \Lambda_0 \) are compact operators on \( L^2_\partial(\Gamma) \) due to the compactness of the trace operator from \( H^1(\Omega) \) into \( L^2(\Gamma) \). To state a factorization of the relative Neumann-to-Dirichlet operator \( \Lambda_0 - \Lambda \), we introduce two auxiliary operators. First, set

\[
H : L^2_\partial(\Gamma) \to L^2(D, \mathbb{R}^d), \quad g \mapsto \nabla u_0|_D
\]

(18)

where \( u_0 \in H^1_\partial(\Omega) \) again solves (17) with \( \gamma = \gamma^0 \). Second, define

\[
T : L^2(D, \mathbb{R}^d) \to L^2(D, \mathbb{R}^d) \quad f \mapsto Q(f - \nabla w),
\]

where \( w \in H^1_\partial(\Omega) \) is a variational solution to

\[
\int_\Omega \gamma \nabla w \cdot \nabla v \, dx = \int_\Omega (\nabla v)^\top Qf \, dx \quad \text{for all } v \in H^1_\partial(\Omega).
\]
Then $\Lambda_0 - \Lambda$ is compact, self-adjoint and positive on $L^2_0(\Gamma)$, see [14, Th. 5.11], and

$$\Lambda_0 - \Lambda = H^*TH \quad \text{on } L^2_0(\Gamma).$$

(19)

Since $\Lambda_0 - \Lambda$ is self-adjoint it possesses an eigendecomposition with eigenvalues $\lambda_j \in \mathbb{R}$ and orthonormal eigenfunctions $\psi_j \in L^2_0(\Gamma)$. The compactness of both $\Lambda_0$ and $\Lambda$ and the positivity of $\Lambda_0 - \Lambda$ shows that $0 < \lambda_j \to 0$ as $j \to \infty$.

The operator $H$ can alternatively be characterized using the so-called Neumann function $\Phi_N(x, y)$ for the domain $\Omega$, see [6, Th. 3.1] for Lipschitz domains and conductivities in $L^\infty_\gamma(\Omega, \mathbb{R}^{d \times d})$ or [19] for domains and conductivities of class $C^{2,\alpha}$. This particular Green’s function for the Neumann problem satisfies for all $y \in \Omega$ that $\Phi_N(\cdot, y) \in H^1(\Omega \setminus \overline{B_r(y)})$ for all $r > 0$, that $\text{div}_x(\gamma_0 \nabla_x \Phi_N(\cdot, y)) = -\delta_y$ in $\Omega$ in the distributional sense, and that $\gamma_0 \nabla \Phi_N(\cdot, y) \cdot \nu = -1/|\Gamma|$ on $\Gamma$. Moreover, $\Phi_N(\cdot, y)$ is Hölder continuous in $\Omega \setminus \{y\}$ and it is symmetric in its two arguments, $\Phi_N(x, y) = \Phi_N(y, x)$ for $x \neq y$ in $\Omega$ (see [6, Eq. (3.2)]).

Finally, due to [6, Eq. (3.4)] the potential

$$u_0(x) = \int_\Gamma \Phi_N(x, y) g(y) dS(y), \quad x \in \Omega, \quad g \in L^2_0(\Gamma),$$

is the unique variational solution in $H^1_0(\Omega)$ to $\text{div}(\gamma_0 \nabla u_0) = 0$ in $\Omega$ and $\gamma_0 \partial u_0/\partial \nu = g$ on $\Gamma$. Hence, we find the following representation of the operator $H$ from (18).

$$Hg = \nabla u_0|_D = \left[\nabla_x \int_\Gamma \Phi_N(x, y) g(y) dS(y)\right]|_D.$$

Given some polarization vector $p \in S^{d-1}$, we set

$$e[z, p](x) = p \cdot \nabla_x \Phi_N(x, z), \quad x \in \Gamma, \quad z \in \Omega.$$

The application of the Factorization method to the impedance tomography problem yields that $e[z, p]$ belongs to the range of $(\Lambda_0 - \Lambda)^{1/2}$ if and only if $z \in D$ (see [14, Theorem 5.14]). Since $\Lambda_0 - \Lambda$ is self-adjoint, the square root $(\Lambda_0 - \Lambda)^{1/2}$ can be defined via the eigendecomposition of this operator and, obviously, equals $[(\Lambda_0 - \Lambda)^* (\Lambda_0 - \Lambda)]^{1/4}$.

Similarly to the Maxwell case, we obtain from the symmetry of the Neumann function with respect to its arguments that

$$p \cdot Hg(z) = p \cdot \nabla_z \int_\Gamma \Phi_N(z, y) g(y) ds(y) = \int_\Gamma p \cdot \nabla_z \Phi_N(y, z) g(y) ds(y) = \langle g, e[z, p]\rangle.$$

Note that Chapter 5.4 in [14] in particular shows that $e[z, p] \in L^2_0(\Gamma)$ for $z \in \Omega$ and $p \in S^{d-1}$.

**Remark 2.** One can analogously apply the Factorization method and the alternative formulation of the Linear Sampling method to shape identification problems for the Stokes-(Brinkman) system in bounded domains, see [17].

### 3 Domain Characterization

In this section, we will present a framework that allows to explain the relation between the Linear Sampling and the Factorization methods for all settings discussed in Section 2.
following definitions and assumptions are motivated by these applications and we expect that other problem classes fit into the same setting as well.

Subsequently, we will assume that \( \Gamma \subset \mathbb{R}^d \) is some open or closed subset of the boundary of a Lipschitz domain; the relative interior of the \( d - 1 \)-dimensional surface \( \Gamma \) is always supposed to be non-empty. We assume further that \( F \) is a closed linear operator defined on a closed subspace \( Y_0 \) of some Hilbert space \( Y \) of \( \mathbb{C}^m \)-valued distributions on \( \Gamma \). Moreover, \( \Omega \) denotes a Lipschitz domain in \( \mathbb{R}^d \).

**Remark 3.** A slightly more general setting where \( F \) operates between dual spaces could also be employed at the expense of a more complicated notation.

In the case of the scalar scattering problems in \( \mathbb{R}^d \) from Section 2.1 \( F \) corresponds to the far field operator from (7), \( \Gamma := \mathbb{S}^{d-1} \), \( Y = Y_0 = L^2(\mathbb{S}^{d-1}) \), and \( \Omega = \mathbb{R}^d \). For the Maxwell problem from Section 2.2 \( F \) corresponds to the far field operator from (15) and we have \( \Gamma := \mathbb{S}^2 \), \( Y = L^2(\mathbb{S}^2, \mathbb{C}^3) \), \( Y_0 = Y_0' = L^2(\mathbb{S}^2) \), and \( \Omega = \mathbb{R}^3 \). Finally, for the impedance tomography problem from Section 2.3 \( F \) corresponds to the relative Neumann-to-Dirichlet operator \( \Lambda_0 - \Lambda \) from (19), \( \Omega \subset \mathbb{R}^d \), \( \Gamma = \partial \Omega \) and \( Y = L^2(\Gamma) \) as well as \( Y_0 = L^2(\Gamma) \). The next assumption links all these measurement operators to obstacles or inclusions inside \( \Omega \) (compare Figure 3.

**Assumption 4.** We assume that the linear operator \( F \) satisfies the following properties:

(A) \( F : Y_0 \to Y_0 \) possesses an orthonormal eigensystem \((\lambda_j, \psi_j)\) with eigenvalues \( \lambda_j \neq 0 \) and eigenvectors \( \psi_j \in Y_0 \), such that \( Fg = \sum_{j \in \mathbb{N}} \lambda_j \langle g, \psi_j \rangle_Y \psi_j \) for all \( g \in Y_0 \). The phases \( \delta_j \) of \( \lambda_j / |\lambda_j| = \exp(i\delta_j) \) belong to some closed interval \( J \subseteq \mathbb{R} \) with length \( |J| < \pi \).

(B) \( F \) possesses a factorization of the form \( F = GH \) for all \( g \in Y_0 \) where \( H \) is a bounded linear operator from \( Y_0 \) into some Banach space \( X \) of \( \mathbb{C}^\ell \)-valued distributions (with \( \ell \in \mathbb{N} \)) defined on a set \( D \subset \Omega \) and \( G : X \to Y_0 \) is closed.

(C) For all \( x \in \Omega \) and \( p \in \mathbb{S}^{\ell-1} \), it holds that \( p \cdot (Hg)(x) = \langle g, e[x,p] \rangle_Y \) for a family \( e[\cdot,\cdot] \in Y_0 \) parametrized by \( x \in \Omega \) and \( p \in \mathbb{S}^{\ell-1} \).

(D) For any \( p \in \mathbb{S}^{\ell-1} \), the function \( e[x,p] \in Y_0 \) belongs to the range of \( (F^*F)^{1/4} : Y_0 \to Y_0 \) if and only if \( x \) belongs to \( D \subset \Omega \).

![Figure 1: Sketch of the framework for shape identification using sampling methods.](image)

From now on, we use the abbreviation

\[
w_g : x \mapsto \langle g, e[x,p] \rangle_Y, \quad x \in \Omega, \quad p \in \mathbb{S}^{\ell-1},
\]

for the operation mapping \( g \in Y_0 \) to the function \( x \mapsto \langle g, e[x,p] \rangle_Y \). Note that the polarization \( p \in \mathbb{S}^{\ell-1} \) is in the sequel arbitrary, but fixed. This is the reason why we do not denote the dependency of \( w_g \) on this parameter explicitly.
Remark 5. (1) For the acoustic scattering problems from Section 2.1, \( w_g \) from (21) is a scalar Herglotz wave function, see [8]. Indeed, \( \ell = 1 \) since the image space \( X \) of the operator \( H \) from [9] contains scalar distributions and hence \( p \in S^0 \) is either plus or minus one. Without loss of generality we choose \( p = 1 \) and set \( e[z](\theta) := e[x,1](\theta) = \exp(-ik\theta \cdot x) \). Since \( Y = L^2(S^2) \),

\[
w_g(x) = \langle g, e[x] \rangle_Y = \int_{S^{d-1}} \exp(ik\theta \cdot x)g(\theta)\,d\theta = v_g(x), \quad x \in \Omega = \mathbb{R}^d.
\]

(2) The situation gets more complicated when considering the electromagnetic scattering problem from Section 2.2. We have \( \ell = 3 \) since \( X = L^2(D,\mathbb{C}^3) \) is a space of vector-valued functions. The function \( w_g \) from (21) turns out to be the dot product of \( p \in S^{\ell-1} = S^2 \) with an electromagnetic Herglotz wave function with density \( g \in Y_0 = L^2(S^3) \),

\[
w_g(x) = \langle g, e[x,p] \rangle_Y = \int_{S^2} \hat{e}[x,p](\theta) \cdot g(\theta)\,dS(\theta) = -ik \int_{S^2} (\theta \times p) \cdot g(\theta)\exp(ik\theta \cdot x)\,dS(\theta) = p \cdot \text{curl} \int_{S^2} \exp(ik\theta \cdot x)g(\theta)\,dS(\theta), \quad x \in \Omega = \mathbb{R}^3.
\]

(3) For the impedance tomography problem from Section 2.3, we have \( \ell = d \) since \( X = L^2(D)^d \). Hence, \( w_g \) from (21) is the directional derivative of a layer potential with density \( g \in Y_0 = L^2(\Gamma) \) in direction \( p \in S^{\ell-1} = S^{d-1} \),

\[
w_g(x) = \langle g, e[x,p] \rangle_Y = \int_{\partial \Gamma} p \cdot \nabla \Phi(x,y)g(y)\,dS(y) = \frac{\partial}{\partial p} \int_{\partial \Gamma} \Phi(x,y)g(y)\,dS(y), \quad y \in \Omega \subset \mathbb{R}^d.
\]

In the introduction we already mentioned the classical formulation of the Linear Sampling method for scalar inverse scattering problems, see [1]. In the abstract framework detailed in Assumption [4], this method can be reformulated as follows: Use the contour lines of the function \( z \mapsto ||g_z^e|| \) where \( g_z^e \) is an approximate solution

\[
Fg_z = e[z,p] \quad \text{in} \quad Y_0 \tag{22}
\]

for parameters \( z \in \Omega \) and \( p \in S^{\ell-1} \) to find the shape of the domain \( D \). Using Tikhonov regularization to tackle the possibly ill-posed operator equation (22) together with the eigen-decomposition \( (\lambda_j, \psi_j)_{j \in \mathbb{N}} \) of \( F \) yields approximate solutions

\[
g_z^e = (\varepsilon I + F^*F)^{-1}F^*e[z,p] = \sum_{j \in \mathbb{N}} \frac{\lambda_j}{|\lambda_j|^2 + \varepsilon} \langle e[z,p], \psi_j \rangle_Y \psi_j, \quad z \in \mathbb{R}^d, \varepsilon > 0. \tag{23}
\]

The regularization parameter \( \varepsilon \) has, again, to be chosen by a parameter choice, e.g., by the discrepancy principle. The claim of the Linear Sampling method is then that the contour lines of \( z \mapsto ||g_z^e|| \) allow to detect the obstacle \( D \) since, for \( y \in D \) there is a better approximation of \( e[z,p] \) in the range of \( F \) than for \( y \notin D \). As mentioned in the introduction there is no rigorous proof for this statement. The basic motivation for the method is a result stating that there exists \( g_{z,\varepsilon} \in Y_0 \) with \( ||Fg_{z,\varepsilon} - e[z,p]|| \leq \varepsilon \) such that for \( z \in D \) and fixed \( \varepsilon > 0 \) it holds that \( ||g_{z,\varepsilon}|| \rightarrow \infty \) as \( z \rightarrow z^* \in \partial D \) while for \( z \notin D \) and \( \varepsilon \rightarrow 0 \) it holds that \( ||g_{z,\varepsilon}|| \rightarrow \infty \) as \( \varepsilon \rightarrow 0 \), see, e.g., [15, 4, 5]. The latter statement should be compared to point (D) of Assumption [4] that provides an exact characterization of \( D \), replacing the range of \( F \) by the range of the square
root \((F^*F)^{1/4}\). For all settings presented in Section 2, this point is precisely the statement of the Factorization method characterizing \(D\) from the measured data \(F\).

Instead of restricting ourselves to Tikhonov regularization, we consider in the sequel any linear regularization scheme \(R_\varepsilon : Y_0 \to Y_0\) defined via a regularizing filter function \(f_\varepsilon\),

\[
R_\varepsilon g := \sum_{j \in \mathbb{N}} f_\varepsilon(|\lambda_j|^2) (F^*g, \psi_j)_Y \psi_j = \sum_{j \in \mathbb{N}} f_\varepsilon(|\lambda_j|^2) \overline{\lambda_j} (g, \psi_j)_Y \psi_j, \quad g \in Y_0.
\]

The (standard) assumptions for the bounded and piecewise continuous filter \(f_\varepsilon : (0, \infty) \to \mathbb{R}\) are

\[
\lim_{\varepsilon \to 0} f_\varepsilon(\lambda) \to \frac{1}{\lambda} \quad \text{for all } \lambda > 0, \quad \lambda |f_\varepsilon(\lambda)| \leq C \quad \text{for all } \varepsilon \geq 0, \lambda > 0.
\]

A classical example for a regularization scheme defined via a filter function that satisfies (25) is Tikhonov regularization. For this scheme, \(f_\varepsilon(\lambda) = 1/(\lambda + \varepsilon)\) and we get as in (23),

\[
g_\varepsilon^T = R_\varepsilon^{\text{Tikh}} \mathbf{e}[z,p] = \sum_{j \in \mathbb{N}} \frac{\overline{\lambda_j}}{|\lambda_j|^2 + \varepsilon} (\mathbf{e}[z,p], \psi_j)_Y \psi_j, \quad g \in Y_0.
\]

Another example is the singular value cut-off with

\[
f_\varepsilon(\lambda) = \begin{cases} 1/\lambda, & |\lambda| \geq \varepsilon, \\ 0, & |\lambda| < \varepsilon. \end{cases}
\]

Here,

\[
g_\varepsilon^{svco} = R_\varepsilon^{\text{svco}} \mathbf{e}[z,p] = \sum_{j : |\lambda_j| \leq \varepsilon} \frac{1}{|\lambda_j|} (\mathbf{e}[z,p], \psi_j)_Y \psi_j, \quad g \in Y_0.
\]

**Theorem 6.** Suppose that Assumption 4 holds, that \(\{R_\varepsilon\}_{\varepsilon > 0}\) is a family of regularization schemes defined via a regularizing filter function, fix \(p \in S^{l-1}\), and define

\[
g_\varepsilon^z := R_\varepsilon \mathbf{e}[z,p] \quad \text{for } z \in \Omega \quad \text{and} \quad \varepsilon > 0.
\]

Then the limit \(\lim_{\varepsilon \to 0} |w_{g_\varepsilon^z}(z)|\) exists if and only if \(z \in D\). For some \(\alpha \in (0,1)\) independent of \(z\) and \(p\),

\[
\alpha \|s_z\|_Y^2 \leq \lim_{\varepsilon \to 0} |w_{g_\varepsilon^z}(z)| \leq \|s_z\|_Y^2,
\]

where \(s_z \in Y\) is the unique solution to \((F^*F)^{1/4}s_z = \mathbf{e}[z,p]\) in \(Y_0\). If \(z \notin D\), then \(|w_{g_\varepsilon^z}(z)|\) tends to infinity as \(\varepsilon \to \infty\) for any \(p \in S^{l-1}\).

**Proof.** The function \(g_\varepsilon^z\) can be explicitly computed as

\[
g_\varepsilon^z = R_\varepsilon \mathbf{e}[z,p] = \sum_{j \in \mathbb{N}} f_\varepsilon(|\lambda_j|^2) \overline{\lambda_j} (\mathbf{e}[z,p], \psi_j)_Y \psi_j
\]

and \(\|g_\varepsilon^z\|_Y^2 = \sum_{j \in \mathbb{N}} |f_\varepsilon(|\lambda_j|^2) \overline{\lambda_j}|^2 |(\mathbf{e}[z,p], \psi_j)_Y|^2\) since \(\psi_j\) is an orthonormal family in \(Y\).

Note that the restriction of the Herglotz wave function \(w_{g_\varepsilon^z}(x)\) to \(D\) equals \(p \cdot H g_\varepsilon^z = (g_\varepsilon^z, \mathbf{e}[x,p])_Y\). The latter is, for fixed \(z \in \Omega\), by Assumption 4(C) a bounded linear form on \(Y_0\).
We can hence interchange this bounded linear form and the series in $j$,

$$w_{\varepsilon}(x) = p \cdot Hg_{\varepsilon}(x) = \sum_{j \in \mathbb{N}} f_{\varepsilon}(|\lambda_j|^2) \overline{\lambda_j} \langle e[z,p], \psi_j \rangle_Y p \cdot H \psi_j(x)$$

$$= \sum_{j \in \mathbb{N}} f_{\varepsilon}(|\lambda_j|^2) \overline{\lambda_j} \langle e[z,p], \psi_j \rangle_Y \langle \psi_j, e[x,p] \rangle_Y$$

$$= \sum_{j \in \mathbb{N}} f_{\varepsilon}(|\lambda_j|^2) \overline{\lambda_j} \langle e[z,p], \psi_j \rangle_Y \langle e[x,p], \overline{\psi_j} \rangle_Y, \quad x \in D.$$

Choosing $x = z$ shows that

$$w_{\varepsilon}(z) = \sum_{j \in \mathbb{N}} f_{\varepsilon}(|\lambda_j|^2) \overline{\lambda_j} \langle e[z,p], \psi_j \rangle_Y^2, \quad z \in D. \quad (29)$$

If $z \in D$, then there exists by Assumption 4[D] a (unique) solution $s_z \in Y_0$ to the equation $(F^*F)^{1/4}s_z = e[z,p]$. Note that

$$s_z = \sum_{j \in \mathbb{N}} \langle e[z,p], \psi_j \rangle_Y \frac{1}{|\lambda_j|^{1/2}} \psi_j, \quad \|s_z\|_Y^2 = \sum_{j \in \mathbb{N}} \frac{|\langle e[z,p], \psi_j \rangle_Y|^2}{|\lambda_j|}, \quad (30)$$

and that the latter norm is finite if and only if $z \in D$ due to Assumption 4[A] and (D) and the well-known Picard criterion. Further,

$$\langle e[z,p], \psi_j \rangle_Y = \langle (F^*F)^{1/4}s_z, \psi_j \rangle_Y = \langle s_z, (F^*F)^{1/4} \psi_j \rangle_Y = |\lambda_j|^{1/2} \langle s_z, \psi_j \rangle_Y.$$

Hence,

$$w_{\varepsilon}(z) = \sum_{j \in \mathbb{N}} f_{\varepsilon}(|\lambda_j|^2) |\lambda_j| \overline{\lambda_j} \langle s_z, \psi_j \rangle_Y^2, \quad z \in D.$$

Note that

$$|w_{\varepsilon}(z)| \leq \sum_{j \in \mathbb{N}} |f_{\varepsilon}(|\lambda_j|^2)| |\lambda_j|^2 \langle s_z, \psi_j \rangle_Y^2 \quad \text{(25)} \leq C \sum_{j \in \mathbb{N}} |\langle s_z, \psi_j \rangle_Y|^2 \leq C \|s_z\|_Y^2, \quad z \in D,$$

where the constant $C$ from (25) is independent of $\varepsilon$. Hence, we can apply the theorem on dominated convergence to deduce that

$$\lim_{\varepsilon \to 0} w_{\varepsilon}(z) = \sum_{j \in \mathbb{N}} \left[ \lim_{\varepsilon \to 0} f_{\varepsilon}(|\lambda_j|^2) \right] |\lambda_j| \overline{\lambda_j} \langle s_z, \psi_j \rangle_Y^2 = \sum_{j \in \mathbb{N}} \overline{\lambda_j} \langle s_z, \psi_j \rangle_Y^2, \quad z \in D.$$

The absolute value of $\lim_{\varepsilon \to 0} w_{\varepsilon}(z)$ is hence bounded from above by

$$\lim_{\varepsilon \to 0} |w_{\varepsilon}(z)| \leq \sum_{j \in \mathbb{N}} |\langle s_z, \psi_j \rangle_Y|^2 = \|s_z\|_Y^2, \quad z \in D.$$

Moreover, by Assumption 4[C], $\lambda_j/|\lambda_j| = \exp(i\delta_j)$ with a phase $\delta_j$ contained in an interval $J$ of length strictly smaller than $\pi$. Choose $\eta \in \mathbb{R}$ such that the shifted interval $J + \eta$ is centered at $\pi/2$, that is, $\text{dist}(J + \eta, 0) = \text{dist}(J + \eta, \pi) =: \delta > 0$. Since $|\exp(i\eta)| = 1$, we can write

$$\lim_{\varepsilon \to 0} |w_{\varepsilon}(z)| = \left| \sum_{j \in \mathbb{N}} \exp(i\delta_j) |\langle s_z, \psi_j \rangle_Y|^2 \right| = \left| \sum_{j \in \mathbb{N}} \exp(i(\delta_j + \eta)) |\langle s_z, \psi_j \rangle_Y|^2 \right|. \quad (31)$$
This choice of \( \eta \) implies that \( \Im \exp(i(\delta_j + \eta)) \geq \sin(\delta) =: \alpha > 0 \) and hence
\[
\lim_{\varepsilon \to 0} |w_{\varepsilon z}(z)| \geq \lim_{\varepsilon \to 0} |\Im w_{\varepsilon z}(z)| \geq \sum_{j \in \mathbb{N}} \Im e^{i(\delta_j + \eta)}|\langle s_z, \psi_j \rangle_Y|^2 \geq \alpha \|s_z\|^2. 
\]

Let now \( z \notin D \). Since the filter function \( f_\varepsilon \) is for fixed \( \varepsilon > 0 \) a bounded, real-valued function, the value of \( w_{\varepsilon z}(z) \) is bounded and its absolute value can be estimated from below using (29) by
\[
|w_{\varepsilon z}(z)| = |e^{-i\eta} w_{\varepsilon z}(z)| = |e^{i\eta} w_{\varepsilon z}(z)| \geq \Im \left(e^{i\eta} w_{\varepsilon z}(z)\right)
= \sum_{j=1}^{\infty} f_\varepsilon(|\lambda_j|^2) \Im \left[e^{i\eta \lambda_j} \langle e[z, p], \psi_j \rangle_Y \right]^2
= \sum_{j=1}^{\infty} f_\varepsilon(|\lambda_j|^2)|\lambda_j| \Im \left[e^{i(\delta_j + \eta)} \langle e[z, p], \psi_j \rangle_Y \right]^2
\]
since, by definition, \( \frac{\lambda_j}{|\lambda_j|} = \exp(i\delta_j) \). Estimating again \( \Im \left[\exp(i(\delta_j + \eta))\right] \geq \sin(\delta) = \alpha \), we obtain, for any \( \varepsilon > 0 \) and any \( N_0 \in \mathbb{N} \),
\[
|w_{\varepsilon z}(z)| \geq \alpha \sum_{j=1}^{N_0} f_\varepsilon(|\lambda_j|^2)|\lambda_j| |\langle e[z, p], \psi_j \rangle_Y|^2.
\]
As \( f_\varepsilon(|\lambda_j|^2)|\lambda_j| \to |\lambda_j|^{-1} \) for \( \varepsilon \to 0 \), taking the limit of the finite sum yields that
\[
\lim_{\varepsilon \to 0} |w_{\varepsilon z}(z)| \geq \alpha \sum_{j=1}^{N_0} |\langle e[z, p], \psi_j \rangle_Y|^2 \left|\frac{\langle e[z, p], \psi_j \rangle_Y}{\lambda_j}\right| 
\text{for arbitrary } N_0 \in \mathbb{N}.
\]
However, since \( z \) does by assumption not belong to \( D \), the function \( e[z, p] \) does by Assumption 4(D) not belong to the range of \( (F^*F)^{1/4} \), i.e., the series \( N_0 \to \sum_{j=1}^{N_0} |\langle e[z, p], \psi_j \rangle_Y|^2 / |\lambda_j| \) grows monotonically without finite upper bound as \( N_0 \to \infty \). In consequence, for any positive zero sequence \( \{\varepsilon_n\}_{n \in \mathbb{N}} \), the sequence \( |w_{\varepsilon_n z}(z)| \) cannot possess any finite accumulation point, that is, \( |w_{\varepsilon z}(z)| \) tends to infinity as \( \varepsilon \to 0 \).

Note that the last proof also shows that for fixed \( \varepsilon > 0 \) and \( z \in \mathbb{R}^3 \) it holds that
\[
\alpha \sum_{j \in \mathbb{N}} f_\varepsilon(|\lambda_j|^2)|\lambda_j|^2 \left|\frac{|\langle e[z, p], \psi_j \rangle_Y|^2}{|\lambda_j|}\right| \leq |w_{\varepsilon z}(z)| \leq \sum_{j \in \mathbb{N}} f_\varepsilon(|\lambda_j|^2)|\lambda_j|^2 \left|\frac{|\langle e[z, p], \psi_j \rangle_Y|^2}{|\lambda_j|}\right|
\leq C \sum_{j \in \mathbb{N}} \left|\frac{|\langle e[z, p], \psi_j \rangle_Y|^2}{|\lambda_j|}\right|
\]
where the constants \( \alpha \) and \( C \) are independent of \( \varepsilon \) and \( z \).

4 Noisy Data and Regularization

In the context of the Linear Sampling method, considering noisy data means considering a perturbed far field operator. Two fundamental problems arise: Firstly, a perturbed far field
operator may fail to be normal and thus the existence of an eigensystem is no longer assured. This means that \( w_{g_z}^{\delta}(z) \) as expressed in [29] will not be computable. Secondly, multiplicity of eigenvalues becomes an issue.

The first problem can be overcome by considering a singular system of \( F \) instead of an eigensystem. Define \( \mu_j = |\lambda_j| \) and \( \varphi_j = (1/\mu_j) F^* \psi_j \). Then \( (\mu_j, \varphi_j, \psi_j) \) is a singular system of \( F \),

\[
Fg = \sum_{j=1}^{\infty} \mu_j \langle g, \varphi_j \rangle \psi_j, \quad g \in Y,
\]

with orthonormal bases \((\varphi_j), (\psi_j)\) and the monotonically decreasing sequence of singular values \( \mu_j \). Also, \( (\mu_j^2, \varphi_j) \) is an eigensystem of \( F^* F \).

Using the singular system, we compute

\[
h_{\varepsilon}(z) = w_{g_z}^{\delta}(z) = \sum_{j=1}^{\infty} f_\varepsilon(|\lambda_j|^2) \bar{\lambda}_j \langle e[z, p], \psi_j \rangle_Y^2 = \sum_{j=1}^{\infty} f_\varepsilon(|\lambda_j|^2) \langle e[z, p], \psi_j \rangle_Y \langle \bar{\lambda}_j \psi_j, e[z, p] \rangle_Y = \sum_{j=1}^{\infty} f_\varepsilon(\mu_j^2) \mu_j \langle e[z, p], \psi_j \rangle_Y \langle \varphi_j, e[z, p] \rangle_Y. \quad (32)
\]

To address the second problem, denote by \((\tilde{\mu}_m)\) the strictly monotonically decreasing sequence of distinct singular values of \( F \) and define \( m_j, j \in \mathbb{N} \), such that \( \mu_{m_j} = \mu_j \). Furthermore, we defined the spectral projections

\[
P_m g = \sum_{j : m_j = m} \langle g, \varphi_j \rangle_Y \varphi_j, \quad m \in \mathbb{N}, \quad g \in Y.
\]

For later use, we note that by orthogonality, we have

\[
\|g\|^2 = \sum_{m=1}^{\infty} \|P_m g\|^2, \quad g \in Y. \quad (33)
\]

Using the spectral projections, we further rewrite the indicator function as

\[
h_{\varepsilon}(z) = \sum_{j=1}^{\infty} f_\varepsilon(\tilde{\mu}_j^2) \mu_j \langle e[z, p], \psi_j \rangle_Y \langle \varphi_j, e[z, p] \rangle_Y = \sum_{j=1}^{\infty} f_\varepsilon(\tilde{\mu}_j^2) \langle F^* e[z, p], \varphi_j \rangle_Y \langle \varphi_j, e[z, p] \rangle_Y
\]

\[
= \sum_{m=1}^{\infty} f_\varepsilon(\tilde{\mu}_m^2) \sum_{j : m_j = m} \sum_{\ell : m_\ell = m} \langle F^* e[z, p], \varphi_j \rangle_Y \langle e[z, p], \varphi_\ell \rangle_Y \langle \varphi_j, \varphi_\ell \rangle_Y
\]

\[
= \sum_{m=1}^{\infty} f_\varepsilon(\tilde{\mu}_m^2) \langle P_m F^* e[z, p], P_m e[z, p] \rangle_Y. \quad (34)
\]

We consider now a noisy far field operator \( F^\delta \) such that \( \|F - F^\delta\|_Y \leq \delta \leq \|F\| \) for some noise level \( \delta \geq 0 \). We then estimate

\[
\| (F^\delta)^* F^\delta - F^* F \| \leq 3 \|F\| \delta.
\]

The perturbed operator \( F^\delta \) gives rise to a perturbed singular system \((\mu_j^\delta, \varphi_j^\delta, \psi_j^\delta)\) and associated projections \( P_m^\delta \) which can be used to compute an approximation \( h_{\varepsilon}^\delta \) of \( h_{\varepsilon} \) by the expression

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We will also assume two bounds on the noise level, i.e.,

\[ \| \mathbf{P}_{\delta}(\mathbf{F})^* \mathbf{e}[z,p], \mathbf{P}_{\delta}^* \mathbf{e}[z,p] \| \gamma . \]

In order to estimate the differences between the expressions for \( h_\varepsilon \) and \( h_\varepsilon^\delta \), we use two lemmas from perturbation theory for self-adjoint operators (see, e.g., Section IV-\$3.1 and Section V-\$4.3 in [12]).

**Lemma 7.** Let \( A, B \) be bounded self-adjoint operators. Then \( \text{dist}(\sigma(A), \sigma(B)) \leq \| A - B \| \), i.e.,

\[ \sup_{\lambda \in \sigma(A)} \text{dist}(\lambda, \sigma(B)), \sup_{\lambda \in \sigma(B)} \text{dist}(\lambda, \sigma(A)) \leq \| A - B \|. \]

**Lemma 8.** Let \( A, B \) be bounded normal operators with monotonically decreasing sequences of distinct eigenvalues \( \{ \lambda^A_m \}_{m \in \mathbb{N}}, \{ \lambda^B_m \}_{m \in \mathbb{N}} \). Denote by \( P^A_m \) and \( P^B_m \) the spectral projections of \( A \) and \( B \), respectively. Assume that \( \| A - B \| \leq \rho < d \) and that \( \text{dist}(\lambda^A_m, \sigma(A) \setminus \{ \lambda^A_m \}) = 2d \) for some \( m \in \mathbb{N} \). If we further set

\[ \tilde{P}^B_m = \sum_{|\lambda^A_m - \lambda^B_n| < d} P^B_n \quad \text{then} \quad \| P^A_m - \tilde{P}^B_m \| \leq \frac{\rho}{d - \rho}. \]

In order to apply these lemmas for obtaining a regularization strategy, relatively large and well separated singular values of \( F \) need to be separated from the rest of the spectrum. We introduce a cut-off index

\[ J(\delta) = \max \left\{ j : \| \mu_j^2, \sigma(F^*F) \setminus \{ \mu_j^2 \} \| > 2(3\| F \| \delta)^{1/3} \right\}. \]

We will also assume two bounds on the noise level \( \delta \): First, \( \delta \leq (6\sqrt{2}\| F \|^{-1}, \text{so that} (3\| F \| \delta)^{2/3} \leq 1/2, \text{and second} \delta \leq \max \{ (\tilde{\mu}_j^2 - \tilde{\mu}_2^2)^2/(24\| F \|), 1 \}, \text{so that} J(\delta) \geq 1 \) (otherwise some of the sums below vanish). Since zero always belongs to the spectrum of the compact operator we note that the definition of \( J(\delta) \) implies

\[ 2(3\| F \| \delta)^{1/3} < \mu_j^2 < \| F \|^2 \quad \text{for all} \ j = 1, \ldots, J(\delta). \]

Obviously, \( J(\delta) \to \infty \) as \( \delta \to 0 \).

To formulate a convergence theorem for a regularized version of the Linear Sampling method for noisy data, let us finally introduce a measure of the variation of the associated filter function,

\[ \text{var}_{\delta'}(f_\varepsilon) = \sup \left\{ \left| f_\varepsilon(\lambda) - f_\varepsilon(\tilde{\lambda}) \right|, 2\delta'^{1/3} \leq \lambda \leq \| F \|^2, |\lambda - \tilde{\lambda}| \leq \delta' \right\}, \quad \delta' > 0. \]

**Theorem 9.** Suppose \( \varepsilon(\delta), 0 < \varepsilon \leq \max \{ (6\sqrt{2}\| F \|)^{-1}, (\tilde{\mu}_j^2 - \tilde{\mu}_2^2)^2/(24\| F \|), 1 \} \) satisfies

\[ \| f_\varepsilon(\delta) \|_\infty \mu J(\delta) \to 0, \quad \| f_\varepsilon(\delta) \|_\infty \sqrt{\delta} \to 0, \quad \text{var}_{\delta'}(f_\varepsilon(\delta)) \to 0, \quad (\delta \to 0). \]

Then, for \( z \in D \),

\[ \lim_{\delta \to 0} \frac{\delta}{h_\varepsilon^\delta(\delta)(z)} = \lim_{\delta \to 0} \frac{\delta}{g_\varepsilon(\delta)}(z). \]
Proof. For \( j \leq J(\delta) \), collect the eigenvalues of \((F^\delta)^* F^\delta\) that are close enough to \( \mu_j^2 \) so that the second lemma above can be applied with \( d = (3\|F\|\delta)^{1/3} > \rho = 3\|F\|\delta \):  

\[
L(m) = \{ \ell : |\tilde{\mu}_m^2 - (\tilde{\mu}_\ell^\delta)^2| \leq 3\|F\|\delta \} ; \quad m \leq m_{J(\delta)} .
\]

Set  

\[
\hat{J}(\delta) = \max \{ \ell : \tilde{m}_\ell \in L(m) \text{ for some } m \leq m_{J(\delta)} \} .
\]

With these definitions there holds

\[
\left( \mu_{\hat{J}(\delta)+1}^\delta \right)^2 \leq \mu_{J(\delta)}^2 - 3\|F\|\delta < \mu_{\hat{J}(\delta)}^2 . \tag{37}
\]

Note also that \( J(\cdot) \) is a strictly monotonically increasing function of \( \delta \).

We estimate from (32) and (35),

\[
|h_\varepsilon(z) - h_\delta^\varepsilon(z)| \leq \sum_{j=1}^{J(\delta)} f_\varepsilon(\mu_j^2) \mu_j \langle e[z,p], \psi_j \rangle_Y \langle \varphi_j, e[z,p] \rangle_Y - \hat{J}(\delta) - \sum_{j=1}^{J(\delta)} f_\varepsilon((\mu_j^2)^2) \mu_j^2 \langle e[z,p], \psi_j \rangle_Y \langle \varphi_j, e[z,p] \rangle_Y
\]

\[
+ \sum_{j=J(\delta)+1}^{\infty} f_\varepsilon(\mu_j^2) \mu_j \langle e[z,p], \psi_j \rangle_Y \langle \varphi_j, e[z,p] \rangle_Y
\]

\[
+ \sum_{j=J(\delta)+1}^{\infty} f_\varepsilon((\mu_j^2)^2) \mu_j^2 \langle e[z,p], \psi_j \rangle_Y \langle \varphi_j, e[z,p] \rangle_Y .
\]

The two series are easily treated. The first can be estimated by

\[
\left| \sum_{j=J(\delta)+1}^{\infty} f_\varepsilon(\mu_j^2) \mu_j \langle e[z,p], \psi_j \rangle_Y \langle \varphi_j, e[z,p] \rangle_Y \right|
\]

\[
\leq \|f_\varepsilon\|_\infty |\mu_{J(\delta)+1}| \sum_{j=J(\delta)+1}^{\infty} |\langle e[z,p], \psi_j \rangle_Y \langle \varphi_j, e[z,p] \rangle_Y| \leq \|f_\varepsilon\|_\infty |\mu_{J(\delta)}| \|e[z,p]\|_Y^2 . \tag{38}
\]

Similarly, we obtain, using (37)

\[
\left| \sum_{j=J(\delta)+1}^{\infty} f_\varepsilon((\mu_j^2)^2) \mu_j^2 \langle e[z,p], \psi_j \rangle_Y \langle \varphi_j, e[z,p] \rangle_Y \right|
\]

\[
\leq \|f_\varepsilon\|_\infty |\mu_{J(\delta)}| \|e[z,p]\|_Y^2 . \tag{39}
\]
Using the representations (34) and (36), the remaining sum is split again into two parts,

\[
\sum_{j=1}^{J(\delta)} f_\varepsilon(\mu_j^2) \mu_j \langle e[z,p], \psi_j \rangle_Y \langle \varphi_j, e[z,p] \rangle_Y - \sum_{j=1}^{J(\delta)} f_\varepsilon((\mu_j^2)^2) \mu_j^4 \langle e[z,p], \psi_j \rangle_Y \langle \varphi_j, e[z,p] \rangle_Y
\]

\[
= \sum_{m=1}^{m_{J(\delta)}} \left[ f_\varepsilon(\mu_m^2) \langle P_m F^* e[z,p], P_m e[z,p] \rangle_Y - \sum_{\ell \in L(m)} f_\varepsilon((\mu_\ell^2)^2) \langle P_\ell^\delta(F^\delta)^* e[z,p], P_\ell^\delta e[z,p] \rangle_Y \right]
\]

\[
\leq \sum_{m=1}^{m_{J(\delta)}} \left| f_\varepsilon(\mu_m^2) \right| \left| \langle P_m F^* e[z,p], P_m e[z,p] \rangle_Y - \sum_{\ell \in L(m)} \langle P_\ell^\delta(F^\delta)^* e[z,p], P_\ell^\delta e[z,p] \rangle_Y \right|
\]

\[
+ \sum_{m=1}^{m_{J(\delta)}} \sum_{\ell \in L(m)} \left| f_\varepsilon(\mu_m^2) - f_\varepsilon((\mu_\ell^2)^2) \right| \left| \langle P_\ell^\delta(F^\delta)^* e[z,p], P_\ell^\delta e[z,p] \rangle_Y \right|.
\]

Using the definition of \(L(m)\), the second sum is seen to be bounded by

\[
\sum_{m=1}^{m_{J(\delta)}} \sum_{\ell \in L(m)} \left| f_\varepsilon(\mu_m^2) - f_\varepsilon((\mu_\ell^2)^2) \right| \left| \langle P_\ell^\delta(F^\delta)^* e[z,p], P_\ell^\delta e[z,p] \rangle_Y \right|
\]

\[
\leq \text{var}_3\|F\|\|\delta\|\sum_{m=1}^{m_{J(\delta)}} \sum_{\ell \in L(m)} \left| \langle P_\ell^\delta(F^\delta)^* e[z,p], P_\ell^\delta e[z,p] \rangle_Y \right|
\]

\[
\leq \text{var}_3\|F\|\|\delta\| \langle (F^\delta)^* e[z,p] \rangle_Y \langle e[z,p] \rangle_Y \leq 2 \text{var}_3\|F\|\|\delta\| \|F\| \|e[z,p]\|_Y^2. \quad (40)
\]

For the first sum, note first that because of orthogonality we have

\[
\sum_{\ell \in L(m)} \langle P_\ell^\delta(F^\delta)^* e[z,p], P_\ell^\delta e[z,p] \rangle_Y = \sum_{\ell, n \in L(m)} \langle P_\ell^\delta(F^\delta)^* e[z,p], P_n^\delta e[z,p] \rangle_Y.
\]

This and Lemma 8 can be applied to obtain

\[
\left| \langle P_m F^* e[z,p], P_m e[z,p] \rangle_Y - \sum_{\ell \in L(m)} \langle P_\ell^\delta(F^\delta)^* e[z,p], P_\ell^\delta e[z,p] \rangle_Y \right|
\]

\[
\leq \left| \langle P_m F^* e[z,p], P_m e[z,p] \rangle_Y - \sum_{\ell \in L(m)} P_\ell^\delta e[z,p] \right| Y
\]

\[
+ \left| \langle P_m F^* e[z,p], \sum_{\ell \in L(m)} P_\ell^\delta e[z,p] \sum_{n \in L(m)} P_n^\delta e[z,p] \right| Y
\]

\[
\leq \left( \|P_m F^* e[z,p]\|_Y + \left\| \sum_{n \in L(m)} P_n^\delta e[z,p] \right\|_Y \right) \frac{3\|F\|\|\delta\|}{(3\|F\|\|\delta\|)^{1/3} - 3\|F\|\|\delta\|}
\]

Standard estimates give

\[
\frac{3\|F\|\|\delta\|}{(3\|F\|\|\delta\|)^{1/3} - 3\|F\|\|\delta\|} = \frac{(3\|F\|\|\delta\|)^{2/3}}{1 - (3\|F\|\|\delta\|)^{2/3}} \leq \frac{(3\|F\|\|\delta\|)^{2/3}}{1 - \frac{1}{2}} = 2 (3\|F\|\|\delta\|)^{2/3}.
\]

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Using the Cauchy-Schwarz inequality and (33), we also obtain
\[
\sum_{m=1}^{m_J(\delta)} \|P_m F^* e[z, p]\|_Y \leq m_J(\delta) \left( \sum_{m=1}^{m_J(\delta)} \|P_m F^* e[z, p]\|_Y^2 \right)^{1/2} \leq \delta^{-1/6} \|F^* e[z, p]\|_Y,
\]
and the corresponding result for \( \sum_{m=1}^{m_J(\delta)} \sum_{n \in L(m)} P_n^* e[z, p] \). Thus, we can estimate
\[
\sum_{m=1}^{m_J(\delta)} |f_\epsilon(\mu_m^2)| \left| \langle P_m F^* e[z, p], P_m e[z, p] \rangle_Y - \sum_{\ell \in L(m)} \langle P_\ell^* (F^\delta)^* e[z, p], P_\ell^* e[z, p] \rangle_Y \right| \leq 2 (3 \|F\|^{2/3} (1 + \|F\|) \|e[z, p]\|_Y \|f_\epsilon\|_\infty \sqrt{\delta}.
\]
Together, the bounds (38)–(41) imply the assertion. \( \square \)

Quite similar arguments also prove that the perturbed indicator function will not remain bounded for a point outside the scatterer.

**Corollary 10.** Assume that \( \delta, \varepsilon(\delta) \) satisfy the conditions of Theorem 9. Then for \( z \notin D \), \( h^\delta_{\varepsilon(\delta)}(z) \) will not remain bounded as \( \delta \to 0 \).

**Proof.** For \( K \in \mathbb{N} \), set
\[
h^K_\varepsilon(z) = \sum_{j=1}^{K} f_\epsilon(\mu_j^2) \mu_j \langle e[z, p], \psi_j \rangle_Y \langle \varphi_j, e[z, p] \rangle_Y,
\]
\[
h^K_{\varepsilon,\delta}(z) = \sum_{j=1}^{K} f_\epsilon(\mu_j^2) \mu_j^\delta \langle e[z, p], \psi_j^\delta \rangle_Y \langle \varphi_j^\delta, e[z, p] \rangle_Y.
\]
Assume there is some constant \( C > 0 \) and some \( \delta_0 > 0 \) such that \( |h^\delta_{\varepsilon(\delta)}(z)| \leq C \) for all \( \delta \in (0, \delta_0) \). We estimate
\[
|h^\delta_{\varepsilon(\delta)}(z)| \leq |h^K_{\varepsilon(\delta)}(z) - h^J_{\varepsilon(\delta)}(z)| + |h^J_{\varepsilon(\delta)}(z) - h^J_{\varepsilon,\delta}(z)| + |h^J_{\varepsilon,\delta}(z) - h^J_{\varepsilon(\delta)}(z)| + C.
\]
However, the three differences can be bounded as in the proof of Theorem 9 using (38), (40) as well as (41) and (39), respectively. Thus, we conclude that \( |h^\delta_{\varepsilon(\delta)}(z)| = |w_{g_\delta(z)}(z)| \) remains bounded as \( \delta \to 0 \), in contradiction to Theorem 6. \( \square \)

**Remark 11.** (a) For specific regularization strategies, the conditions on the behaviour \( \varepsilon(\delta) \) given in Theorem 9 take on a more concrete form. Considering Tikhonov regularization as in (20), for example, we can write the first condition as \( \mu_{j(\delta)}/\varepsilon(\delta) \to 0 \) (\( \delta \to 0 \)) whereas the second and third conditions both follow from \( \delta^{1/2}/\varepsilon(\delta) \to 0 \) (\( \delta \to 0 \)). In the case of the spectral value cut-off as in (27), the first condition follows again from \( \mu_{j(\delta)}/\varepsilon(\delta) \to 0 \) (\( \delta \to 0 \)) and the second from \( \delta^{1/2}/\varepsilon(\delta) \to 0 \) (\( \delta \to 0 \)) whereas the third condition only requires \( \delta/\varepsilon(\delta) \to 0 \) (\( \delta \to 0 \)).

(b) Note that the regularization scheme requires information about the singular values of \( F \) to determine the parameter choice \( \varepsilon(\delta) \). Due to Lemma 7 we know that the Hausdorff distance of the singular values of the data \( F \) and \( F^\delta \) is as small as \( \| (F^\delta)^* F^\delta - F^* F \| \leq 2 \| F^\delta \| + \delta^2 \). Hence, at least for small noise level \( \delta \), replacing the singular values of \( F \) by those of \( F^\delta \) yields a sufficiently accurate approximation.
5 Numerical Experiments

We illustrate the theoretical results with some examples for the electromagnetic inverse shape identification problem introduced in Section 2.2. The special case under consideration is when the dielectricity \( \varepsilon \) equals a constant \( \varepsilon_1 \) inside the scatterer \( D \) that differs from the background value \( \varepsilon_0 \) taken outside \( \overline{D} \). Assuming that a plane incident wave \( H^i \) of the form \( \| \) scatters from the dielectric body \( D \), the scattering problem reduces to a transmission problem for the total and scattered magnetic fields \( H \) and \( H^s \) in \( D \) and \( \mathbb{R}^3 \setminus \overline{D} \), respectively,

\[
\begin{align*}
\text{curl} \text{curl} H^s - k^2 H^s &= 0 \quad \text{in} \ \mathbb{R}^3 \setminus \overline{D}, \\
\text{curl} \left[ \frac{1}{\varepsilon_r} \text{curl} H \right] - k^2 H &= 0 \quad \text{in} \ D,
\end{align*}
\]

\[
\left\{ \begin{array}{l}
\nu \times (H^s - H) = \nu \times H^i \\
\nu \times \text{curl} \left( H^s - \frac{1}{\varepsilon_r} H \right) = \nu \times \text{curl} H^i
\end{array} \right. \quad \text{on} \ \partial D,
\]

together with the Silver-Müller radiation condition for \( H^s \) at infinity. This transmission problem can be analytically tackled using boundary integral equations \([3]\) and numerically discretized and using boundary element methods. Our numerical experiments rely on scattering data computed using the boundary element package BEM++, see \([21]\). Computing the discretized and using boundary element methods. Our numerical experiments rely on scattering data computed using the boundary element package BEM++, see \([21]\). Computing far field patterns for incident plane waves of the form \( \| \) with uniformly distributed directions \( \{ \theta_j \}_{j=1}^{2N} \subset \mathbb{S}^2 \) and two associated orthogonal polarizations \( \{ p_j^{(1)}, p_j^{(2)} \}_{j=1}^{2N} \subset \mathbb{S}^2 \) such that \( p_j^{(1,2)} \cdot \theta_j = 0 \), we obtain a far field matrix of size \( 2N \times 2N \). Due to the unavoidable discretization error, this discrete far field matrix due not contain exact values of \( H^\infty(\theta, \theta_j; p_j^{(1,2)}) \) but approximations with a certain numerical error \( \delta'(N) > 0 \). Using suitable interpolation projections one shows that this matrix is a natural discretization of an approximation \( F_N \) to the far field operator \( F \) from \([15]\) that converges in the operator norm as \( N \to \infty \) if the numerical error \( \delta'(N) \) arising in the computation of the far fields \( H^\infty(\theta, \theta_j; p_j^{(1,2)}) \) tends to zero as \( N \to \infty \), see \([16]\).

In the subsequent numerical experiments we provide examples for the reconstruction of two dielectric scatterers at wave number \( k = 5\pi/4 \) from a far field matrix \( F_N \) of size \( 336 \times 336 \), that is, involving far field data for \( N = 168 \) uniformly distributed directions on the sphere. The numerical computation of the far field patterns uses a surface integral equation of the first kind for the exterior Cauchy data of the magnetic fields on the Lipschitz boundary \( \partial D \) of the scatterer \( D \). The integral equation is derived from the Stratton-Chu representation formula (see eq. (48) in \([3]\)) and posed in the product Sobolev space \( H^{-1/2}_\infty(\text{curl} \partial D, \partial D) \times H^{-1/2}_\infty(\text{curl} \partial D, \partial D) \). Here, \( H^{-1/2}_\infty(\text{curl} \partial D, \partial D) \) is the trace space for functions in \( H(\text{curl}, D) \) (see \([3]\) for details). We discretize this integral equation using a Galerkin boundary element method in the lowest order Raviart-Thomas spaces using the BEM++-Python interface on a Linux workstation with 32 CPU cores and 128 GB RAM. The two dielectric scatterers we consider in the example below are, first, a prism with triangular base and height one, see Figure 2(a). The corner points of the lower base are \( (1,1,1)^T, (2,1,1)^T, \) and \( (1,2,1)^T \), such that the three remaining top corners of the prism are \( (1,1,2)^T, (2,1,2)^T, \) and \( (1,2,2)^T \). Second, we consider a scatterer consisting of two balls with diameter 1 centered at \( (0,0,0)^T \) and \( (2,0,0)^T \), see Figure 2(b). For both dielectrics, the constant relative permittivity inside the scatterer equals 1.3. The surface mesh of the prism contains 37620 surface triangles while the mesh of the balls contains 17056 surface triangles. We finally remark that the polarization vector \( p \) used in the test function \( e[z, p] \) equals \( p = (1,1,1)^T/\sqrt{3} \) in all examples.

For this computational setting, the above-described far-field matrix \( F_N \) possesses a normality error \( \| F_N^* F_N - F_N^* F_N \|_2 \) of about 0.27 and 0.22 for the first and second dielectric
scatterer, respectively. This synthetically computed data is hence far from being highly accurate but, as the reconstructions below will indicate, is sufficient to obtain shape information on the two scattering objects via all three methods, with somewhat distinct quality, however. Since the synthetic scattering data contains noise we regularize all three methods: The factorization method is stabilized by not plotting the series from (30) (that can anyway not be computed since the data matrix $F_N$ is not normal and in general fails to possess eigenvalues). Instead, we use the singular value decomposition $(\mu_j, \varphi_j, \psi_j)_{j=1}^{2N}$ of $F_N$ and evaluate the test-function $e[z,p]$ at the set of $N = 168$ uniformly distributed directions on the sphere used for the forward computations, and project into the tangent space. This yields a vector $e_N[z,p]$ of size $2N$ and allows to plot

$$z \mapsto \sum_{j=1}^{2N} \frac{|\langle e_N[z,p], \phi_j \rangle_{C^{2N}}|^2}{|\mu_j| + \varepsilon} \quad \text{for a parameter } \varepsilon > 0.$$

For the relatively high noise level in our data, such a regularization turned out to be crucial to obtain reasonable reconstructions. For the Linear Sampling method we plot the norm of the Tikhonov regularization $g_N = g_N(z, \varepsilon) \in C^{2N}$, solution to

$$(F_N^* F_N + \varepsilon I_N)g_N = F_N^* e_N[z,p] \quad \text{in } C^{2N}$$

as a function of $z$. This is the usual way the Linear Sampling method is implemented. Indeed, using a discrepancy principle pointwise for each $z$ to determine the parameter $\varepsilon$ is costly and does, according to our experiments, not improve the reconstructions. The vector $g_N = g_N(z, \varepsilon)$ hence plays the role of the density $g^z$ in the above theoretic statements. In consequence, to obtain an indicator function via the alternative formulation of the Linear Sampling method presented in Theorem 6, we evaluate a discretization of the Herglotz wave function $w_{g_1}(z)$ following (28),

$$z \mapsto \sum_{j=1}^{2N} \beta_j g_N(z, \varepsilon)(j)e_N[z,p](j).$$
The weights $\beta_j > 0$ are derived from a quadrature rule on the sphere; in our case, the $N = 168$ directions are uniformly distributed and all weights equal $4\pi/168$. Note that the parameters $\varepsilon$ are chosen independently for all three methods, that all plots below are computed on uniform three dimensional Cartesian grids with mesh width $0.1$, and that we scale all images by the maximal value such that the color scales of all two-dimensional slice plots equal each other.

Figure 3 shows reconstructions for the dielectric prism from Figure 2(a). For each method, an isosurface with level $c > 0$ as well as a slice of the indicator function are shown. With the factorization method, we obtain an accurate reconstruction of the location of the obstacle. The shape itself is not reconstructed as accurately, with edges and corners being smoothed out. With both variants of the Linear Sampling method, the location is equally well reconstructed, but the overall shape is much more blurry. Arguably, the alternative Linear Sampling method gives a better separation of points inside and outside the obstacle and standard Linear Sampling does.

Figure 4 shows reconstructions for the dielectric medium from Figure 2(b) formed by two balls. The quality of the results are quite similar to those obtained in Figure 3 and the same remarks apply.

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References


Figure 3: Reconstructions of the dielectric prism from Figure 2(a). (a) Factorization method, $\varepsilon = 0.2$, $c = 0.60$. (b) Factorization method, $\varepsilon = 0.2$, $c = 0.60$, slice at $z = 1.5$. (c) Linear Sampling method, $\varepsilon = 1 \cdot 10^{-7}$, $c = 0.4$. (d) Linear Sampling method, $\varepsilon = 1 \cdot 10^{-7}$, $c = 0.4$, slice at $z = 1.5$. (e) Alternative Linear Sampling method, $\varepsilon = 4 \cdot 10^{-7}$, $c = 0.4$. (f) Alternative Linear Sampling method, $\varepsilon = 4 \cdot 10^{-7}$, $c = 0.4$, slice at $z = 1.5$. 
Figure 4: Reconstructions of the two balls from Figure 2(b). (a) Factorization method, $\varepsilon = 0.3$, $c = 0.62$. (b) Factorization method, $\varepsilon = 0.3$, $c = 0.62$, slice at $y = 0$. (c) Linear Sampling method, $\varepsilon = 0.001$, $c = 0.65$. (d) Linear Sampling method, $\varepsilon = 0.001$, $c = 0.65$, slice at $y = 0$. (e) Alternative Linear Sampling method, $\varepsilon = 0.003$, $c = 0.65$. (f) Alternative Linear Sampling method, $\varepsilon = 0.003$, $c = 0.65$, slice at $y = 0$. 


