A Simple Method for Solving Inverse Scattering Problems in the Resonance Region

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Abstract
This paper is concerned with the development of an inversion scheme for two dimensional inverse scattering problems in the resonance region which does not use nonlinear optimization methods and is relatively independent of the geometry and physical properties of the scatterer. It is assumed that the far field pattern $u_\infty (\varphi; \theta)$ corresponding to observation angle $\varphi$ and plane waves incident at angle $\theta$ is known for all $\varphi, \theta \in [-\pi, \pi]$. From this information, the support of the scattering obstacle is obtained by solving the integral equation

$$
\int_{-\pi}^{\pi} u_\infty (\varphi; \theta) g(\theta) \, d\theta = e^{-ik\rho \cos(\varphi-\alpha)}, \quad \varphi \in [-\pi, \pi],
$$

where $k$ is the wave number and $y_0 = (\rho \cos \alpha, \rho \sin \alpha)$ is on a rectangular grid containing the scatterer. The support is found by noting that $\|g\|_{L^2([-\pi, \pi])}$ is unbounded as $y_0$ approaches the boundary of the scattering object from inside the scatterer. Numerical examples are given showing the practicality of this method.

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1 Introduction

Inverse scattering problems for frequencies in the resonance region are particularly difficult to solve numerically since it is no longer possible to use various linear approximation methods (e.g. the Born or physical optics approximation). This usually means that some type of nonlinear optimization scheme is used which is not only computationally expensive but also suffers from possible local minima. In addition, for inverse obstacle scattering, one needs to know the type of boundary condition (Dirichlet, Neumann, etc.) as well as geometric information about the scatterer (e.g. connected and starlike with respect to the origin) both of which may in fact be unknown. Nevertheless, for many problems of practical importance one is forced to use frequencies in the resonance region in order to achieve sufficient resolution. Hence, it would be highly desirable to develop rapid methods for solving inverse scattering problems in the resonance region which do not require nonlinear optimization methods for their resolution and are also only mildly dependent on the geometry and physical properties of the scatterer. In order to have hopes of achieving such a goal, it is of course necessary to lower one’s expectations, i.e. one has to be willing to make some sacrifices concerning accuracy and detail. For example, if the physical properties of the scatterer are unknown then the support of the scatterer is probably the best that one can hope for in any given inversion procedure.

The purpose of this paper is to initiate the study of inversion schemes for inverse scattering problems in the resonance region which fulfill the above criteria. We will only assume that the scatterer is a finite set of domains $D_n, n = 1, 2, ..., N$, in the plane $\mathbb{R}^2$ bounded by smooth Jordan curves $\partial D_n, n = 1, 2, ..., N$, and make no assumption on the physical nature of the scatterer (i.e. whether or not the waves penetrate into $D_n$ and, if they don’t, what type of boundary condition is satisfied by the total field on $\partial D_n$). Of course, as mentioned above, for such a general scattering problem we can only hope to determine the support of each $D_n$. A further drawback of our method is that it is based on knowing the far field pattern for many angles of incidence and observation distributed uniformly around the scattering object and it is assumed the host medium is non-absorbing. However, we view this paper as just the beginning in a long term study of similar inversion scheme and our hope is to eventually be able to relax some of the above assumptions. (For example, implementing our method for scatterers in $\mathbb{R}^3$ would be one such step).

In this paper, for the sake of simplicity, we will consider as our model the case of the scattering of electromagnetic plane waves by an infinite cylinder with (bounded) cross section $\mathcal{D}$ and with the electric field polarized parallel to the cylinder. We assume that the boundary $\partial \mathcal{D}$ is twice continuously differentiable and make no assumption on whether or not the total electric field penetrates the cylinder and if it doesn’t what type of boundary condition (i.e. Dirichlet, Neumann or impedance) is satisfied by the total field on the boundary. Under the above assumptions, the total field $u$ satisfies

$$\Delta_2 u + k^2 u = 0 \text{ in } \mathbb{R}^2 \setminus \mathcal{D}$$

(1.1)
\[ u(x) = e^{i k x \cdot d} + u^s(x) \]  
\[ \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0 \] 

where \( k > 0 \) is the wave number, \( r = |x| \), \( d \) is the angle of incidence of the incident wave, \( |d| = 1 \), and the radiation condition (1.3) for the scattered field \( u^s \) is assumed to hold uniformly for all directions. We will denote the polar coordinates for \( x \) and \( d \) by \( x = (r \cos \varphi, r \sin \varphi) \) and \( d = (\cos \theta, \sin \theta) \) respectively. From (1.1) – (1.3) we have that (c.f.[1])

\[ u^s(r, \varphi) = \frac{e^{i k r}}{\sqrt{r}} u_\infty(\varphi; \theta) + \mathcal{O} \left( \frac{1}{r^{3/2}} \right) \]  

where \( u_\infty \) is the far field pattern. The inverse scattering problem we want to solve is to determine the support of \( D \) from a knowledge of \( u_\infty \) for \( \varphi, \theta \in [-\pi, \pi] \).

The plan of our paper is as follows. Let \( \epsilon > 0 \) and \( y_0 = (\rho \cos \alpha, \rho \sin \alpha) \in D \). Then we will show that for scattering by either an obstacle or a penetrable media there exists a function \( g = g(\cdot; y_0) \in L^2(-\pi, \pi) \) such that

\[ \int_{-\pi}^{\pi} u_\infty(\cdot; \theta) g(\theta; y_0) \, d\theta = e^{-ik\rho \cos(-\alpha)} + \mathcal{O} \left( \frac{1}{r^{3/2}} \right) \]  

and that as \( y_0 \to \partial D \), \( y_0 \in D \), we have

\[ \lim_{y_0 \to \partial D} \| g(\cdot; y_0) \|_{L^2(-\pi, \pi)} = \infty. \]  

Furthermore, if we define the Herglotz wave function \( v_g \) with kernel \( g(\cdot; y_0) \) by

\[ v_g(r, \varphi) = \int_{-\pi}^{\pi} e^{i k r \cos(\varphi - \theta)} g(\theta; y_0) \, d\theta, \]  

we have that

\[ \lim_{y_0 \to \partial D} \max_{x \in \partial D} \{|v_g(\cdot; y_0)(x)| + |\nabla v_g(\cdot; y_0)(x)|\} = \infty. \]  

These results suggest the following simple method for finding \( \partial D \). First, find a (regularized) solution \( g = g(\cdot; y_0) \) of the far field equation

\[ \int_{-\pi}^{\pi} u_\infty(\varphi; \theta) g(\theta; y_0) \, d\theta = e^{-ik\rho \cos(\varphi - \alpha)}, \quad \varphi \in [-\pi, \pi], \]

for \( y_0 = (\rho \cos \alpha, \rho \sin \alpha) \) on some partition \( P \) of a rectangle known a priori to contain the scattering obstacle. (Equations (1.7) – (1.9) suggest the appropriate regularization method is to require \( \| g' \| \) to be bounded). Having found \( g = g(\cdot; y_0) \) in this manner,
\( \partial D \) is determined by those points \( y_0 \in P \) where \( \| g( \cdot ; y_0 ) \| \) achieves its maximum. Numerical examples of the implementation of this approach will be presented in the last section of this paper.

We note that the above approach has some resemblance to the methods used by Isakov ([5]) and Kirsch and Kress ([6]) to prove uniqueness theorems for the inverse scattering problem.

2 The Far Field Equation.

We first consider the case of scattering by an obstacle, i.e. on \( \partial D \) the solution of (1.1) – (1.3) satisfies either the Dirichlet boundary condition

\[
u = 0 \text{ on } \partial D, \tag{2.1a}
\]
or the Neumann boundary condition

\[
\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D \tag{2.1b}
\]

where \( \nu \) is the unit outward normal to \( \partial D \), or the impedance boundary condition

\[
\frac{\partial u}{\partial \nu} + i \lambda u = 0 \text{ on } \partial D \tag{2.1c}
\]

where \( \lambda > 0 \). For the existence and uniqueness of a solution to (1.1) – (1.3), (2.1) see [1]. Suppose one of these three boundary conditions is satisfied. Fix \( y_0 = (\rho \cos \alpha, \rho \sin \alpha) \).

Note that \( e^{-ik\rho \cos (\varphi - \alpha)} \) is the far field pattern of

\[
e^{i\pi/4} \sqrt{\frac{\pi k}{2}} H_0^{(1)}(k|x - y_0|) \tag{2.2}
\]

where \( H_0^{(1)} \) is a Hankel function of the first kind and order zero. Assume that \( y_0 \in D \). Then, if \( g \) satisfies the far field equation (1.9), by Rellich’s Lemma ([1]) we have that

\[
\int_{-\pi}^{\pi} u^s(r, \varphi; \theta) g(\theta) \, d\theta = e^{i\pi/4} \sqrt{\frac{\pi k}{2}} H_0^{(1)}(k|x - y_0|) \tag{2.3}
\]

for \( x = (r \cos \varphi, r \sin \varphi) \in \mathbb{R}^2 \setminus D \). We observe that the left hand is the scattered wave corresponding to the incident field \( v_g \). Letting \( x \) tend to \( \partial D \) and using the Dirichlet, Neumann- or impedance boundary condition we conclude that equation (1.9) is solvable in \( L^2(-\pi, \pi) \) if and only if there exists a solution of the boundary value problem

\[
\Delta w + k^2 w = 0 \text{ in } D \tag{2.4}
\]

and

\[
w(x) + e^{i\pi/4} \sqrt{\frac{\pi k}{2}} H_0^{(1)}(k|x - y_0|) = 0 \text{ on } \partial D \tag{2.5a}
\]
for the Dirichlet boundary condition or
\[
\frac{\partial}{\partial \nu} \left[ w(x) + e^{i\pi/4} \sqrt{\frac{\pi k}{2}} H_0^{(1)}(k|x-y_0|) \right] = 0 \text{ on } \partial D \quad (2.5b)
\]
for the Neumann boundary condition or
\[
\left( \frac{\partial}{\partial \nu} + i\lambda \right) \left[ w(x) + e^{i\pi/4} \sqrt{\frac{\pi k}{2}} H_0^{(1)}(k|x-y_0|) \right] = 0 \text{ on } \partial D \quad (2.5c)
\]
for the impedance boundary condition, and \( w \) is a Herglotz wave function of the form
\[
w(x) = v_g(x) = \int_{-\pi}^{\pi} e^{ikr\cos(\varphi-\theta)} g(\theta) \, d\theta \quad (2.6)
\]
for some \( g \in L^2(-\pi, \pi) \). Therefore, equation (1.9) is solvable only in very special situations (e.g. if \( D \) is a disk with origin \( y_0 \), cf. Example 2.2 below). Let us assume from now on that \( k^2 \) is neither a Dirichlet or Neumann eigenvalue for \( D \). Then there exists a unique solution \( w \) of the boundary value problem (2.4), (2.5) (which may not have an extension to a Herglotz function in \( \mathbb{R}^2 \)). However, it is known ([19]) that for every \( \epsilon > 0 \) there exists \( g \in L^2(-\pi, \pi) \) such that
\[
\max_{x \in D} |v_g(x) - w(x)| \leq \epsilon \text{ and } \max_{x \in D} |\nabla v_g(x) - \nabla w(x)| \leq \epsilon. \quad (2.7)
\]
Furthermore, since \( u_\infty \) depends continuously on the boundary data, we can even conclude that for every \( \epsilon > 0 \) there exists \( g \in L^2(-\pi, \pi) \) such that also (1.5) holds, i.e.
\[
\left\| \int_{-\pi}^{\pi} u_\infty(\cdot; \theta) g(\theta) \, d\theta - e^{-ik\rho\cos(\alpha)} \right\|_{L^2(-\pi, \pi)} < \epsilon. \quad (2.8)
\]
Letting \( x \) tend to the boundary and using the boundary condition we conclude that
\[
\max_{x \in \partial D} \left| v_g(x) + e^{i\pi/4} \sqrt{\frac{\pi k}{2}} H_0^{(1)}(k|x-y_0|) \right| < \epsilon \quad (2.9a)
\]
for the Dirichlet problem,
\[
\max_{x \in \partial D} \left| \frac{\partial}{\partial \nu} v_g(x) + e^{i\pi/4} \sqrt{\frac{\pi k}{2}} H_0^{(1)}(k|x-y_0|) \right| < \epsilon \quad (2.9b)
\]
for the Neumann problem and
\[
\max_{x \in \partial D} \left| \left( \frac{\partial}{\partial \nu} + i\lambda \right) v_g(x) + e^{i\pi/4} \sqrt{\frac{\pi k}{2}} H_0^{(1)}(k|x-y_0|) \right| < \epsilon \quad (2.9c)
\]
for the impedance problem. Now we indicate the dependence on \( y_0 \) again by writing 
\[ g(\cdot; y_0) \text{ and } v(\cdot; y_0) = v_g(\cdot; y_0). \]

We can now establish the following theorem:

**Theorem 2.1** Assume that \( D \) is simply connected, \( k^2 \) is neither a Dirichlet or Neumann eigenvalue and that on \( \partial D \) the solution \( u \) of (1.1) – (1.3) satisfies one of the boundary conditions (2.1). Then for every \( \epsilon > 0 \) and \( y_0 \in D \) there exists a function 
\[ g(\cdot; y_0) \in L^2(\pm \pi) \] such that (2.8) is satisfied,
\[
\lim_{y_0 \rightarrow \partial D} \| g(\cdot; y_0) \|_{L^2(\pm \pi)} = \infty
\]
and, if \( v(\cdot; y_0) \) is the Herglotz wave function with kernel \( g(\cdot; y_0) \),
\[
\lim_{y_0 \rightarrow \partial D} \max_{x \in \partial D} \{ |v(x; y_0)| + |\nabla v(x; y_0)| \} = \infty.
\]

**Proof:** Let 
\[
v(x; y_0) = \int_{-\pi}^{\pi} e^{-ik\cos(\phi-\theta)} g(\theta; y_0) d\theta
\]
be a Herglotz wave function which satisfies (2.8) and (2.9).

From (2.9) we have that \( |v(x; y_0)|, |\frac{\partial}{\partial \theta} v(x; y_0)| \) or \( |(\frac{\partial}{\partial x} + i\lambda) v(x; y_0)| \) must become unbounded for \( x \in \partial D \) and \( y_0 \) tending to \( x \). But this implies that \( |v(x; y_0)| + |\nabla v(x; y_0)| \) becomes unbounded as \( y_0 \) tends to \( x \in \partial D \). Suppose that \( \|g(\cdot; y_0)\|_{L^2} \) remains bounded as \( y_0 \) tends to a point \( x \in \partial D \). Then from (2.10) and the Schwarz inequality \( |v(x; y_0)| + |\nabla v(x; y_0)| \) is bounded which we have just seen is not true. Hence \( \|g(\cdot; y_0)\|_{L^2} \) becomes unbounded as \( y_0 \) tends to \( \partial D \). \( \square \)

The following example shows that \( \|g(\cdot; y_0)\|_{L^2} \) may also be large for points inside \( D \), i.e. the boundary may not be sharply defined. This is also evident in some of the numerical examples in the following section. When this ambiguity exists, it can often be resolved through the use of a priori information or further information contained in the far field data. For example, if it is known that the scatterer is an imperfect conductor with given surface impedance, then a lower bound on the arc length can be obtained from the far field data and this can help to remove the ambiguity (2).

**Example 2.2** Consider the scattering problem (1.1) – (1.3), (2.1a) when \( D \) is a disk of radius \( a \) centered at the origin and assume that \( k^2 \) is not a Dirichlet eigenvalue. Then
\[
u_{\infty}(\varphi; \theta) = -e^{-i\omega/4} \sqrt{\frac{2}{\pi k^2}} \sum_{n=-\infty}^{\infty} \frac{J_n(k\alpha)}{H_n^{(1)}(k\alpha)} e^{i\omega(\varphi-\theta)}
\]
(2.11)
where \( J_n \) is the Bessel function of order \( n \) and \( H_n^{(1)} \) is the Hankel function of the first kind and order \( n \). In this case, writing
\[
g(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{i\omega \theta}
\]
(2.12)
and using the Jacobi-Anger expansion to equate Fourier coefficients in (1.9), we see that
\[ a_n = \frac{i^{-n} H_n^{(1)}(ka) J_n(kp)}{2\pi J_n(ka)} e^{-in\alpha}, \]
(2.13)
i.e. \( g \in L^2(-\pi, \pi) \) only if \( \rho = 0 \). If \( \rho \neq 0 \) a solution \( g \in L^2(-\pi, \pi) \) to (1.9) does not exist, but (1.5) is satisfied by
\[ g_N(\theta) = \sum_{-N}^{N} a_n e^{in\theta} \]
for \( N \) sufficiently large. From (2.6) we see that the Herglotz wave function \( v_N \) having \( g_N \) as its kernel is given by
\[ v_N(r, \varphi) = \sum_{-N}^{N} \frac{J_n(kr) H_n^{(1)}(ka) J_n(kp)}{J_n(ka)} e^{in(\varphi-\alpha)}. \]
(2.15)
The of the theorem is given by \( g = g_N \) where \( N \) tends to infinity as \( \rho \to a \). From the asymptotic behaviour of Bessel and Hankel functions for large \( n \), we see that, as \( \rho \to a \), \( |v_N(a, \varphi)| \) and \( \|g_N\|_{L^2} \) indeed tend to infinity. However, \( \|g_N\|_{L^2} \) is also large for \( \rho < a \). For a numerical example in the case when \( a = 1 \) see the next section of this paper.

We note that the above theorem says that for every \( \epsilon > 0 \) there exists \( g \in L^2(-\pi, \pi) \) such that (1.5) is satisfied. However, in practice we need to determine \( g \) from the integral equation of the first kind (1.9) and there is no guarantee that the \( g \) so obtained is in fact the \( g \) of the theorem. This difficulty is dealt with by noting that the highly oscillatory components of \( g \) (e.g. the higher order Fourier coefficients) cause only small changes in both the right hand side of (1.9) and the Herglotz wave function \( v \) having \( g \) as its kernel. In particular, we can assume that a \( g \) satisfying the theorem has these components filtered out. As mentioned in the Introduction, this suggests regularizing the integral equation (1.9) by using a penalty term of the form \( \gamma\|g\|_{L^2}^2 \) where \( \gamma \) is a regularization parameter (c.f. [4]). This assures that if \( \epsilon \) is sufficiently small the \( g \) obtained by solving (1.9) in this manner is close in the \( L^2 \)-norm to the \( g \) whose existence is established by the above theorem. We hope to provide a more detailed analysis of these ideas in future work.

We now turn our attention to the problem when \( D \) is no longer an obstacle but a (penetrable) inhomogeneous medium. Let \( c = c(x) \) be the local speed of sound and let
\[ n(x) = \frac{c_0^2}{c(x)^2} \left( 1 + i \frac{\gamma(x)}{\omega} \right) \quad \text{and} \quad m(x) = 1 - n(x), \]
(2.16)
where \( \omega \) is the frequency and \( c_0 \) the speed of sound in vacuum. \( \gamma(x) \) describes the absorption of the medium. Define \( D := \{ x \in \mathbb{R}^2 : n(x) \neq 1 \} \). We make the assumptions that \( n \in C^2(D) \), the region \( D \) is simply connected with \( C^4 \)-boundary \( \partial D \).
and, furthermore, that there exists $c > 0$ with $|m(x)| \geq c$ for all $x \in \overline{D}$. The latter condition implies that $n$ has a jump across $\partial D$.

Equation (1.1) is now replaced by

$$\Delta_2 u + k^2 n(x) u = 0 \text{ in } \mathbb{R}^2. \quad (2.17)$$

In this case, the existence of a solution to (1.9) is equivalent to the existence of a function $w \in C^2(D) \cap C^1(\overline{D})$ and Herglotz wave function $v$ with kernel $g$ such that $w$ and $v$ satisfy the interior transmission problem ([1], [8])

$$\begin{align*}
\Delta_2 w + k^2 n(x) w &= 0 \quad \text{in } D \\
\Delta_2 v + k^2 v &= 0 \\
w - v &= c H_0^{(1)}(k|x - y_0|) \\
\frac{\partial}{\partial 
u}(w - v) &= c \frac{\partial}{\partial 
u} H_0^{(1)}(k|x - y_0|) \quad \text{on } \partial D
\end{align*} \quad (2.18)$$

where

$$c = e^{i \pi / 4} \sqrt{\frac{\pi k}{2}}. \quad (2.20)$$

The role of Dirichlet and Neumann eigenvalues for the case of obstacle scattering is now replaced by transmission eigenvalues and to avoid these eigenvalues we make the assumption that $\text{Im } n(x) > 0$ for some $x \in D$ (c.f. [8]). We can now prove the following theorem for scattering by an inhomogeneous medium that is the analogue of the theorem proven above for obstacle scattering.

**Theorem 2.3** Assume that $D$ is connected, $k^2$ is not a Dirichlet eigenvalue for $D$, $\text{Im } n(x) > 0$ for some $x \in D$ and let $u$ be the solution of (2.17), (1.2) and (1.3). Then for every $\epsilon > 0$ and $y_0 \in D$ there exists a function $g(\cdot; y_0) \in L^2(-\pi, \pi)$ such that (1.5) is satisfied,

$$\lim_{y_0 \to \partial D} \|g(\cdot; y_0)\|_{L^2(-\pi, \pi)} = \infty$$

and, if $v(\cdot; y_0)$ is the Herglotz wave function with kernel $g(\cdot; y_0)$,

$$\lim_{y_0 \to \partial D} \max_{x \in \partial D} |v(x; y_0)| = \infty.$$

**Proof:** From the results in [8] which are valid in $\mathbb{R}^2$ as well as in $\mathbb{R}^3$ we know that for every $y_0 \in D$ there exists a solution $\tilde{v}, \tilde{w} \in H^2_{\text{loc}}(D) \cap L^2(D)$ such that $\tilde{v} - \tilde{w} \in H^2(D)$ and equations (2.18), (2.19) are satisfied. Here, $H^2_{\text{loc}}(D) = \{ u : D \to \mathbb{C} : u|_U \in H^2(U) \text{ for every open set } U \text{ with } \overline{U} \subset D \}$. The boundary conditions are understood in the sense of the trace theorem for functions in $H^2(D)$. Note that $\tilde{v}$ is, in general, not a Herglotz wave function.

Now define

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|), \quad x \neq y. \quad (2.21)$$
Then by Green's formula and the trace theorem we have that for \( x \in D \)
\[
\tilde{w}(x) = \tilde{v}(x) - k^2 \int_D \Phi(x, y) \, m(y) \, \tilde{w}(y) \, dy \\
+ \int_{\partial D} \left[ [\tilde{v}(y) - \tilde{w}(y)] \frac{\partial}{\partial \nu(y)} \Phi(x, y) - \Phi(x, y) \frac{\partial}{\partial \nu} [\tilde{v}(y) - \tilde{w}(y)] \right] \, ds(y).
\]
\[ \tag{2.22} \]

Furthermore, for \( x \in D \) and \( y_0 \in D \) we have that
\[
\int_{\partial D} \left\{ H_0^{(1)}(k|y - y_0|) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial H_0^{(1)}(k|y - y_0|)}{\partial \nu(y)} \Phi(x, y) \right\} \, ds(y) = 0. \tag{2.23} \]

Hence, adding (2.22) and (2.23) and using the boundary conditions (2.19), we conclude that for \( x, y_0 \in D \)
\[
\tilde{w}(x) = \tilde{v}(x) - k^2 \int_D \Phi(x, y) \, m(y) \, \tilde{w}(y) \, dy, \quad x \in D. \tag{2.24} \]

Note that \( \tilde{w} \) and \( \tilde{v} \) depend also on \( y_0 \in D \). By Theorem 10.3 of [1] (which is valid for piecewise continuous \( n(x) \) and for \( x \in \mathbb{R}^2 \) as well as for \( x \in \mathbb{R}^3 \)), for every \( \epsilon_1 > 0 \) we can find solutions \( g = g(\cdot; y_0) \in L^2(S^1) \) and \( w = w(\cdot; y_0) \in H^2(D) \) such that the Herglotz wave function \( v_g \) with kernel \( g \) and \( w \) solve (2.18) and
\[
\left\| w - v - c H_0^{(1)}(k \cdot -y_0) \right\|_{L^2(\partial D)} < \epsilon_1 \tag{2.25a} \]
\[
\left\| \frac{\partial}{\partial \nu} (w - v - c H_0^{(1)}(k \cdot -y_0)) \right\|_{L^2(\partial D)} < \epsilon_1. \tag{2.25b} \]

By choosing \( \epsilon_1 \) sufficiently small, we can guarantee that (1.5) is satisfied (c.f. equation (10.19) of [1]) and, furthermore, that
\[
\left\| \tilde{w} - w \right\|_{L^2(D)} + \left\| \tilde{v} - v_g \right\|_{L^2(D)} \leq \epsilon. \tag{2.26} \]

The latter follows from the fact that the solution of the interior transmission problem (2.18), (2.19) depends continuously on the boundary data (c.f. [8]). Again we note that also \( w \) and \( g \) depend on \( y_0 \in D \). Therefore, from now on we write \( w(\cdot; y_0) \) and \( v(\cdot; y_0) = v_g(\cdot; y_0) \) and analogously for \( \tilde{w} \) and \( \tilde{v} \) to indicate this dependence.

Suppose that, in contradiction to the statement of the theorem, \( \max_{x \in \partial D} |v(x; y_0)| \) is bounded as \( y_0 \) tends to \( \partial D \). Then, since \( k^2 \) is not a Dirichlet eigenvalue, \( v \) depends continuously on its boundary data, i.e. we can assume that \( \max_{x \in \partial D} |v(x; y_0)| \leq M_1 \) for all \( y_0 \in D \).

Estimate (2.26) implies that also \( \left\| \tilde{v}(\cdot; y_0) \right\|_{L^2(D)} \leq M_2 \) for all \( y_0 \in D \). Now we consider (2.24) as an integral equation for \( \tilde{w} \) (c.f. Theorem 8.7 of [1]) and conclude that also \( \left\| \tilde{w}(\cdot; y_0) \right\|_{L^2(D)} \) is bounded independently of \( y_0 \in D \). Since the volume
potential is bounded from $L^2(D)$ into $H^2(D)$ we conclude from (2.24) that even 
$\| \tilde{w}(\cdot, y_0) - \tilde{v}(\cdot, y_0) \|_{H^2(D)}$ is bounded independently of $y_0 \in D$. The trace theorem yields that 
$\| \tilde{w}(\cdot, y_0) - \tilde{v}(\cdot, y_0) \|_{H^{3/2}(\partial D)}$ is bounded, i.e. there exists $M_3 > 0$ with 
\[
\| H_0^{(1)}(k|\cdot - y_0|) \|_{H^{3/2}(\partial D)} \leq M_3 
\]
for all $y_0 \in D$. This is impossible since $H_0^{(1)}$ has a logarithmic singularity, and $H^{3/2}(\partial D)$ is continuously imbedded in $C(\partial D)$. Therefore, we have proven that $\| v(\cdot, y_0) \|_{L^2(\partial D)}$ is unbounded as $y_0$ tends to $\partial D$. The fact that $\| g \|_{L^2}$ is unbounded as $y_0$ tends to $\partial D$ now follows as in the theorem for obstacle scattering. \hfill $\square$

**Remark:** The proof has to be modified in the case of $\mathbb{R}^3$. The Hankel function $\nu H_0^{(1)}(k|x - y_0|)$ has to be replaced by the fundamental solution $\exp \left( ik|x - y_0| / |x - y_0| \right)$. Now $H^{3/2}(\partial D)$ is no longer continuously imbedded in $C(\partial D)$. Instead, we use the argument that the function $x \mapsto \exp \left( ik|x - y_0| / |x - y_0| \right)$ is not bounded in the $L^2(\partial D)$-norm independently of $y_0 \in D$.

### 3 Numerical Examples

We will now present several numerical examples illustrating the applicability of the above method for solving the inverse scattering problem. In particular, we will consider problems corresponding to the scattering of plane waves with respect to Dirichlet, Neumann and impedance boundary conditions and one problem associated with the scattering of plane waves by a penetrable inhomogeneous medium. For the cases of obstacle scattering, the synthetic data are generated by using the method of boundary integral equations (c.f. Section 3.5 of [1]) whereas for the case of scattering by an inhomogeneous medium we use a coupled finite element and boundary element method ([7]). To solve the inverse problem, we solve the (discretized) far field equation (1.9) and then plot the contours of the mapping 
\[
y_0 \mapsto \log \| g(\cdot, y_0) \|_{L^2}
\]
for $y_0$ on a rectangular grid containing the scatterer. For all examples we have taken the wave number $k = 1$.

Figure 1 shows the contour for the unit circle with Dirichlet boundary conditions. Figures 2 and 3 show the case of the "single kite" parametrized by 
\[
x(t) = (1.5 \sin(t), \cos(t) + 0.65 \cos(2t) - 0.65), \quad 0 \leq t \leq 2\pi,
\]
(cf. p. 70 of [1]) for Dirichlet- (Fig. 2, right), Neumann- (Fig. 3, left) and impedance boundary condition, (Fig. 3, right. Here we have chosen impedance $\lambda = 1$).
In Figure 4 we show the results for the "double kite" (i.e. the single kite plus a second kite rotated 45° and displaced from the first kite along the vector (5, 5)) with respect to Dirichlet boundary conditions. We have chosen 50 contour lines. In Figure 5 the same example was taken, but for the contour lines of values 5, 5.2, ..., 7 (left) and values 6 and 6.2 (right). To our knowledge this is the first method which is able to determine two obstacles without knowing a priori the number of obstacles.

In each of these cases we have computed the far field pattern $u_{\infty}(\varphi; \theta)$ at 32 equidistantly distributed observation points $\varphi_j = j \cdot \pi/32$, $j = 0, \ldots, 31$, and 32 incident directions $\theta_j = j \cdot \pi/32$, $j = 0, \ldots, 31$, for wave number $k = 1$. We have chosen grids for $y_0$ of $61 \times 61$ points in every case.

Figure 6 shows the result for the scattering by an inhomogeneous medium. We have used the "twin peaks" example considered in [3], i.e.

$$n(x_1, x_2) = \begin{cases} 
\frac{1}{2} (3 + \cos 2\pi r_1) & \text{if } r_1 < 0.5, \\
\frac{1}{2} (3 + \cos 2\pi r_2) & \text{if } r_2 < 0.5, \\
1 & \text{otherwise,}
\end{cases} \quad (3.27)$$

where $r_1 = \left( (x_1 - \frac{1}{2})^2 + x_2^2 \right)^{1/2}$ and $r_2 = \left( (x_1 + \frac{1}{2})^2 + x_2^2 \right)^{1/2}$. We would like to thank Peter Monk for computing this example. Here we have used 51 incident waves and determined $g$ by least squares best fit for $g$ a trigonometric polynomial of degree 11 (i.e. 23 Fourier coefficients).

4 References


