The Factorization Method for a Class of Inverse Elliptic Problems
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Abstract
In this paper the factorization method from inverse scattering theory and impedance tomography is extended to a class of general elliptic differential equations in divergence form. The inverse problem is to determine the interface \( \partial \Omega \) of an interior change of the material parameters from the Neumann-Dirichlet map. Since absorption is allowed a suitable combination of the real and imaginary part of the Neumann-Dirichlet map is needed to explicitly characterize \( \Omega \) by the data.

1 Introduction
The factorization method is a fairly new approach for solving classes of inverse problems where an internal interface \( \Gamma \) has to be determined from measurements of the Cauchy data on the surface of a body \( B \) containing \( \Gamma \) in its interior.

Historically, this method was first developed in [13], [14] for inverse scattering problems. Here, the shape \( \Gamma = \partial \Omega \) of the scattering object \( \Omega \) has to be determined by the far field pattern \( u^\infty(\hat{x},\hat{\theta}) \) of the scattering field in direction \( \hat{x} \) for all directions \( \hat{\theta} \) of incident plane waves. The method has been extended to crack problems in [17]. Inspired by these papers, Brühl and Hanke studied factorization methods in impedance tomography in [1, 2, 3]. The original version of this method required that the scattering matrix had to be unitary – or, in the case of impedance tomography, – that the Neumann-Dirichlet operator had to be selfadjoint and positive (or negative) definite. This prevented applications with absorption. Recently, new versions of the factorization method have been developed in [15, 16, 10, 9] which can handle problems with absorption as well. It is the aim of this paper to extend this approach to elliptic equations of the form

\[
\begin{align*}
\text{div} (\gamma \nabla u) + qu &= 0 \quad \text{in some region } B, \\
\partial_\gamma u &= f \quad \text{on } \partial B,
\end{align*}
\]

and wish to obtain information about \( \gamma \) from the knowledge of \( (f, u|_{\partial B}) \) for all \( f \). Here, for \( x \in \partial B \) and smooth functions \( u \) the co-normal derivative \( \partial_\gamma u|_{\partial B} \) of \( u \) at \( x \) is defined by \( \partial_\gamma u(x) = n(x) \cdot (\gamma(x) \nabla u(x)) \), where \( n = n(x) \), \( x \in \partial B \), denotes the exterior unit normal vector at \( x \in \partial B \). The coefficient \( \gamma \) is space dependent and matrix valued. The weak formulation of the boundary value problem will be recalled in Section 1. The problem (1.1) defines the Neumann-Dirichlet operator \( \Lambda \) which assigns to each function \( f \) on \( \partial B \) the Dirichlet trace \( u|_{\partial B} \) and, of course depends on \( \gamma \) and \( q \). Therefore, we can rephrase the inverse problem to determine information about \( \gamma \) and \( q \) from the knowledge of \( \Lambda \). In the following, we describe three possible applications of this problem.
(A) Impedance Tomography:
The first problem we have in mind is the problem of impedance tomography. On the surface of a body $B$ one applies a set of known currents and measures the corresponding voltages on the same surface. From this information one tries to recover the internal conductivity – or at least the shape of the region where the conductivity differs from a known background situation. Applications of impedance tomography include medical imaging and non destructive testing. The mathematical model for the coupling of currents and voltages is given by the quasi-static Maxwell’s equations
\[ \text{curl} \ H = (-i\omega \varepsilon + \sigma) E, \quad \text{curl} \ E = 0, \quad (1.2) \]
which leads to the potential $u$ through $E = \nabla u$ and, taking the divergence of the first equation, to
\[ \text{div} (\gamma \nabla u) = 0 \quad \text{in} \ B \quad \text{where} \quad \gamma = \sigma - i\omega \varepsilon. \quad (1.3) \]
On the boundary $\partial B$ the current and voltage distribution are given by $n \cdot (\gamma E) = \partial \gamma u$ and $u|_{\partial B}$, respectively. Therefore, we are led to problem (1.1) for $q \equiv 0$.

(B) Inverse Scattering Theory in $\mathbb{R}^3$:
Time harmonic (acoustic) waves of the form
\[ v^{inc}(x,t) = \text{Re} \left[ a \exp \left( \frac{i\omega |x-y|/c_0 - i\omega t}{4\pi |x-y|} \right) \right], \quad x \neq y, \]
with frequency $\omega > 0$ and speed $c_0$ are scattered by a medium which is characterized by a density distribution $\rho = \rho(x)$ and speed of sound $c = c(x)$. The linearized wave equation is
\[ \frac{\partial^2 v(x,t)}{\partial t^2} = c(x)^2 \rho(x) \text{div} \left[ \frac{1}{\rho(x)} \nabla v(x,t) \right]. \]
Outside of some region $\Omega$ the parameters $\rho$ and $c$ are constant with $c \equiv c_0$ and $\rho \equiv \rho_0$. The total wave is of the form $v(x,t) = \text{Re} \left[ a v(x) e^{-i\omega t} \right]$ and satisfies the reduced wave equation
\[ \text{div} \left[ \frac{\rho_0}{\rho(x)} \nabla v(x) \right] + \frac{\omega^2 \rho_0}{c(x)^2 \rho(x)} v(x) = 0. \quad (1.4) \]
Therefore, in this situation $\gamma = \rho_0/\rho$ and $q = (\omega^2 \rho_0)/(c^2 \rho)$. Outside of $\Omega$ equation (1.4) reduces to the Helmholtz equation $\Delta v + k^2 v = 0$ with wave number $k = \omega/c_0$. The Sommerfeld radiation has to be imposed on $v^s = v - \Phi(\cdot, y)$. Here $\Phi$ denotes the fundamental solution in $\mathbb{R}^3$, given by
\[ \Phi(x, y) := \frac{\exp(ik|x-y|)}{4\pi |x-y|}, \quad x \neq y. \quad (1.5) \]
We write $v = v(x, y)$ to indicate the dependence on the source point $y$. The inverse problem is to determine $\rho$ and/or $c$ from the knowledge of $v(x, y)$ for all $x, y \in \partial B$ where $B$ is some domain which contains $\overline{\Omega}$ in its interior. We note that $v = v(x, y)$ is just the Green’s function of the differential equation. From the knowledge of $v(x, y)$ for $x, y \in \partial B$
we can compute the Neumann-Dirichlet map $\Lambda : H^{-1/2}(\partial B) \to H^{1/2}(\partial B)$. Indeed, let $h \in H^{-1/2}(\partial B)$. We make an ansatz for $u$ in the form of the single layer potential

$$u(x) = \int_{\partial B} \psi(y) v(x,y) \, ds(y), \quad x \in B.$$ 

Then $u$ solves the differential equation (1.4) in $B$, and $\partial u/\partial n = f$ on $\partial B$ holds if and only if $\psi$ solves the integral equation

$$\frac{1}{2} \psi(x) + \int_{\partial B} \psi(y) \frac{\partial v(x,y)}{\partial n(x)} \, ds(y) = f(x), \quad x \in \partial B. \quad (1.6)$$

This follows from the jump conditions for the traces of the double layer potential.\footnote{Equation (1.6) is a Fredholm equation of the second kind. Uniqueness holds provided the Neumann problem in $B$ has a unique solution for all $f \in H^{-1/2}(\partial B)$. This is seen by standard arguments. Therefore, (1.6) is uniquely solvable for any $f \in H^{-1/2}(\partial B)$, and $\Lambda$ can be expressed by

$$\Lambda = S \left( \frac{1}{2} I + K \right)^{-1}$$

where $K$ and $S$ are the extensions of the operators

$$K \psi(x) = \int_{\partial B} \psi(y) \frac{\partial v(x,y)}{\partial n(x)} \, ds(y), \quad x \in \partial B,$$

$$S \psi(x) = \int_{\partial B} \psi(y) v(x,y) \, ds(y), \quad x \in \partial B,$$

respectively. Therefore, at least from a mathematical point of view inverse scattering problems are also subsumed under our problem.

\textbf{(C) Thermal Imaging:} This is also a technique in nondestructive testing and evaluation. A heat source is used to produce a heat flux at the boundary of an object and the resulting temperature distribution on the surface is measured. Assuming that the heat flux is time harmonic, i.e. of the form $\text{Re} \left[ f(x) e^{-i\omega t} \right]$ with frequency $\omega > 0$, the model is formulated by the boundary value problem

$$\text{div} \left( \kappa \nabla u \right) + i \omega c \rho u = 0 \quad \text{in } B,$$

$$\kappa \frac{\partial u}{\partial n} = f \quad \text{on } \partial B,$$

where $\kappa$, $c$, $\rho$ denote the thermal conductivity, specific heat, and density, respectively. The inverse problem is again, to determine internal interfaces from additional measurements of the temperature on $\partial B$. We refer to [4] for a related problem. Therefore, we are again led to problem (1.1) with $\gamma = \kappa$ and $q = i \omega c \rho$.

\footnote{Here we use already the mapping properties of the Neumann-Dirichlet map between Sobolev spaces $H^{\pm 1/2}(\partial B)$.}

\footnote{Note that $v$ is as singular as the fundamental solution $\Phi$ - at least in a neighborhood of $\partial B$.}
The paper is organized as follows. In Section 2 we will study in detail the case where \( q \) vanishes. The approach consists of three parts explained in Subsections 2.2, 2.3, and 2.4. First, the difference of the Neumann-Dirichlet operators for the perturbed and unperturbed cases is factorized in the form \( \Lambda - \Lambda_0 = G(T - T_0)G^* \) where \( G \) is “more explicit” than \( \Lambda - \Lambda_0 \) and the real part of \( T - T_0 \) is coercive. In Subsection 2.3 it is shown using arguments from spectral theory that the ranges of \( G \) and \( |\text{Re} \Lambda - \Lambda_0|^{1/2} \) coincide. Therefore, using an eigensystem of \( \text{Re} \Lambda - \Lambda_0 \) the range of \( G \) can explicitly be characterized. In Subsection 2.4 we construct functions \( \varphi_y \) depending on \( y \in \mathbb{R}^n \) with the property that \( y \) belongs to the support \( \Omega \) of the contrast \( \gamma - \gamma_0 \) if, and only if, \( \varphi_y \) belongs to the range of \( G \). Combining the previous two results yields an explicit characterization of \( \Omega \) by the eigensystem of \( \text{Re} \Lambda - \Lambda_0 \). Although some of the results carry over easily to the more general case of arbitrary \( q \) we prefer this approach also to make the exposition better readable.

In Section 3 the same program is carried over to the case of general \( q \). Here we will use compact perturbation arguments to characterize the range of \( G \) by \( \Lambda - \Lambda_0 \). We will not prove those results of this section which are only slight modifications of the corresponding results of Section 2.

2 The Factorization Method in Electrical Impedance Tomography for Anisotropic Materials

2.1 The Problem of Impedance Tomography

Let \( B \subset \mathbb{R}^n, n = 2 \) or 3, denote a bounded simply connected domain with \( C^2 \)-boundary \( \partial B \). Let \( \gamma : B \to \mathbb{C}^{n \times n} \) be a matrix valued complex function. Having the quasi-static approximation of Maxwell’s equations in mind we think of \( \gamma \) being of the form \( \gamma = \sigma - i\omega \varepsilon \) with conductivity \( \sigma \), frequency \( \omega \), and permittivity \( \varepsilon \). Given a current distribution \( f \) on \( \partial B \) the potential is determined from the boundary value problem

\[
\text{div} \left( \gamma \nabla u \right) = 0 \quad \text{in} \ B, \quad \partial \gamma u = f \quad \text{on} \ \partial B. \tag{2.1}
\]

Of course, smoothness assumptions on \( \gamma \) have to be imposed in order to give a meaning to this problem (see below). We consider (2.1) as a (not necessarily small) perturbation of the problem

\[
\text{div} \left( \gamma_0 \nabla u_0 \right) = 0 \quad \text{in} \ B, \quad \partial \gamma_0 u_0 = f \quad \text{on} \ \partial B. \tag{2.2}
\]

We formulate our basic assumption \((A1)\) on \( \gamma_0 \) and \( \gamma \) as follows. Further assumptions will be added below.

\((A1)\) Let \( \gamma_0 \in C^{2,\alpha}(\overline{B}) \) be real, symmetric and uniformly positive definite, i.e. there exists \( c_0 > 0 \) with

\[
z^\ast \gamma_0(x) z \geq c_0 |z|^2 \quad \text{for all} \ z \in \mathbb{C}^n \ \text{and all} \ x \in B.
\]

We assume that \( \gamma \) has the form

\[
\gamma(x) = \begin{cases} 
\gamma_0(x), & x \in B \setminus \Omega, \\
\gamma_0(x) + \gamma_1(x), & x \in \Omega,
\end{cases}
\]
where \( \gamma_1 \in L^\infty(\Omega) \) is complex valued and symmetric (not hermitean!) a.e. on \( \Omega \). Here, \( \Omega \subset \mathbb{R}^n \) denotes a domain with \( C^2 \)-boundary \( \partial \Omega \) such that \( \overline{\Omega} \subset B \) and \( B \setminus \overline{\Omega} \) is connected. We assume, furthermore, that there exists \( c_1 > 0 \) with

\[
\Re \left[ z^* \gamma(x) z \right] \geq c_1 |z|^2 \quad \text{and} \quad \Im \left[ z^* \gamma(x) z \right] \leq 0 \quad \text{for all} \ z \in \mathbb{C}^n \ \text{and almost all} \ x \in \Omega.
\]

In the formulation of this assumption we have used the matrix notation, i.e. we consider vectors \( z \in \mathbb{C}^n \) as \( n \times 1 \)-matrices and denote by \( z^* = \bar{z}^\top \) the (complex) adjoint of \( z \).

Before we recall the correct setting of the problem we formulate the inverse problem: One tries to recover the support \( \Omega \) of \( \gamma_1 \) from measurements of all pairs \( (f, u_f|_{\partial B}) \) where we indicated the dependence of \( u \) on \( f \) by writing \( u_f \).

Let \( H^1(B) \) denotes the classical Sobolev (Hilbert-)space of order one, \( H^{1/2}(\partial B) \) the space of traces \( u_{|_{\partial B}} \) for \( u \in H^1(B) \), and \( H^{-1/2}(\partial B) \) its dual space. By \( \langle f, g \rangle \) we denote both, the inner product in \( L^2(\partial B) \) and its extension, i.e. the dual form in \( \langle H^{-1/2}(\partial B), H^{1/2}(\partial B) \rangle \).

Furthermore, from the divergence theorem in the form

\[
0 = \int_B \operatorname{div}(\gamma \nabla u) \, dx = \int_{\partial B} \partial_\nu u \, ds
\]

we observe that existence can only hold if the boundary data satisfy \( \int_{\partial B} f \, ds = 0 \) or, in weak form, \( \langle f, 1 \rangle = 0 \) where \( 1 \) denotes the constant function with value one. Therefore, we introduce the closed subspaces \( H^1_\diamond(B) \) and \( H^{1/2}_\diamond(\partial B) \) of \( H^1(B) \) and \( H^{1/2}(\partial B) \), respectively, by

\[
H^{1/2}_\diamond(\partial B) = \{ f \in H^{1/2}(\partial B) : \langle f, 1 \rangle = 0 \},
\]

\[
H^1_\diamond(B) = \{ u \in H^1(B) : u_{|_{\partial B}} \in H^{1/2}_\diamond(\partial B) \}.
\]

For \( f \in H^{-1/2}_\diamond(\partial B) \) the solutions \( u, u_0 \in H^1_\diamond(B) \) of (2.1) and (2.2), respectively, are understood in the variational sense, i.e.

\[
\int_B \nabla \varphi(x)^* \gamma(x) \nabla u(x) \, dx = \langle f, \varphi \rangle \quad \text{for all} \ \varphi \in H^1_\diamond(B), \quad (2.3)
\]

and analogously for \( u_0 \). It is well known that the theorem of Lax-Milgram yields uniqueness and existence of solutions of (2.3) for every \( f \in H^{-1/2}_\diamond(\partial B) \).

The data \( (f, u_f|_{\partial B}) \) for the inverse problem are collected in the Neumann-Dirichlet operator \( \Lambda : H^{-1/2}_\diamond(\partial B) \to H^{1/2}_\diamond(\partial B) \), defined by \( \Lambda f = u_f|_{\partial B} \) where \( u_f \) solves (2.1) in the sense of (2.3). Analogously, by \( \Lambda_0 \) we denote the operator for the unperturbed case, i.e. for (2.2) instead of (2.1). The question of uniqueness in impedance tomography is the problem whether the operator \( \Lambda \) determines the admittance \( \gamma \) uniquely. While this is the case for scalar functions \( \gamma \) simple arguments using a change of variables show that a matrix valued function \( \gamma \) is, in general, not uniquely determined. The support \( \Omega \), however, is determined by \( \Lambda \) under certain conditions on \( \gamma_0 \) and \( \gamma_1 \) (see [11] for an overview on uniqueness results).
2.2 The Factorization of $\Lambda - \Lambda_0$

The basis of the method is the following factorization of the operator $\Lambda - \Lambda_0$. We define the operators

$$G : H^{-1/2}_0(\partial \Omega) \to H^{1/2}_0(\partial B) \quad \text{and} \quad T : H^{1/2}_0(\partial \Omega) \to H^{-1/2}_0(\partial \Omega)$$

as follows:

For $\psi \in H^{-1/2}_0(\partial \Omega)$ we set $G\psi = v|_{\partial B}$ where $v \in H^1(B \setminus \overline{\Omega})$ with $v|_{\partial B} \in H^{-1/2}_0(\partial B)$ solves the boundary value problem in $B \setminus \overline{\Omega}$:

$$\begin{align*}
\text{div} \left( \gamma_0 \nabla v \right) &= 0 \text{ in } B \setminus \overline{\Omega}, \quad \partial_\gamma v = \psi \text{ on } \partial \Omega, \quad \partial_\gamma v = 0 \text{ on } \partial B , \quad (2.4)
\end{align*}$$

For $h \in H^{1/2}_0(\partial \Omega)$ we set $Th = \partial_\gamma w_+$ on $\partial \Omega$ where $w \in H^1(B \setminus \overline{\Omega}) \cap H^1(\Omega)$ with $w|_{\partial B} \in H^{1/2}_0(\partial B)$ solves the boundary value problem

$$\begin{align*}
\text{div} \left( \gamma \nabla w \right) &= 0 \text{ in } B \setminus \partial \Omega, \quad \partial_\gamma w = 0 \text{ on } \partial B , \quad (2.5a) \\
\partial_\gamma w_+ - \partial_\gamma w_- &= 0 \text{ on } \partial \Omega, \quad w_+ - w_- = h \text{ on } \partial \Omega . \quad (2.5b)
\end{align*}$$

Here and in the following we denote by $w_\pm$ the trace of $w$ from the exterior and interior, respectively. The variational form of (2.5) is to find $w \in H^1(B \setminus \overline{\Omega}) \cap H^1(\Omega)$ with $w|_{\partial B} \in H^{1/2}_0(\partial B)$ and $w_+ - w_- = h$ such that

$$\int_B \nabla \varphi^* \gamma \nabla w \, dx = 0 \quad \text{for all } \varphi \in H^1_0(B) . \quad (2.6)$$

By $T_0$ we denote the operator $T$ when $\gamma$ is replaced by $\gamma_0$. We note that $T$ and $T$ are even defined and bounded as operators from $H^{1/2}(\partial \Omega)$ to $H^{-1/2}_0(\partial \Omega)$.

**Theorem 2.1** The following factorization holds:

$$\Lambda - \Lambda_0 = G(T - T_0)G^* . \quad (2.7)$$

Furthermore, the operator $T_0$ is selfadjoint, and the adjoint $T^* : H^{1/2}_0(\partial \Omega) \to H^{-1/2}_0(\partial \Omega)$ of $T$ is given by $T^*h = \partial_\gamma w_+$ on $\partial \Omega$ where $w \in H^1(B \setminus \overline{\Omega}) \cap H^1(\Omega)$ with $w|_{\partial B} \in H^{1/2}_0(\partial B)$ solves the boundary value problem

$$\begin{align*}
\text{div} \left( \overline{\gamma} \nabla w \right) &= 0 \text{ in } B \setminus \partial \Omega, \quad \partial_\gamma w = 0 \text{ on } \partial B , \quad (2.8a) \\
\partial_\gamma w_+ - \partial_\gamma w_- &= 0 \text{ on } \partial \Omega, \quad w_+ - w_- = h \text{ on } \partial \Omega . \quad (2.8b)
\end{align*}$$

In particular, $T$ is selfadjoint if $\gamma$ is real valued.

**Proof:** First, we prove that $T^*$ has the given form. Indeed, let $Th = \partial_\gamma w_+ = \partial_\gamma w_-$ and let $v$ be the solution of (2.8) for $g$ instead of $h$. By Green’s theorem we have

$$\begin{align*}
\langle Th, g \rangle - \langle h, \partial_\gamma v_+ \rangle &= \int_{\partial \Omega} \{ \partial_\gamma w_+ [\overline{v}_+ - v_-] - [w_+ - w_-] \partial_\gamma \overline{v}_+ \} \, ds \\
&= \int_{\partial B} \{ v \partial_\gamma w - w \partial_\gamma v \} \, ds - \int_{\partial \Omega} \{ v_+ \partial_\gamma w_+ - v_- \partial_\gamma w_- \} \, ds = 0
\end{align*}$$
which proves the desired form of $T^*$.

By the definitions of $\Lambda - \Lambda_0$ and $G$ we observe that $(\Lambda - \Lambda_0)\varphi = (u - u_0)|_{\partial B} = G\psi$ where $\psi = \partial_{\gamma_0}(u - u_0)|_{\partial B}$. We introduce the auxiliary operator $L : H^{1/2}_0(\partial B) \to H^{1/2}_0(\partial \Omega)$ by $Lf = \partial_{\gamma_0}u_+$ where $u$ solves (2.1). Analogously, $L_0$ is defined. Then we observe that $\Lambda - \Lambda_0 = G(L - L_0)$. By two applications of Green’s theorem as above it is easily shown that the adjoint $L^* : H^{1/2}_0(\partial \Omega) \to H^{1/2}_0(\partial B)$ of $L$ is given by $L^*h = w|_{\partial B}$ where $w$ solves (2.8). Therefore, $L^*h = w|_{\partial B} = GT^* h$. Analogously, $L^*_0 = GT^*_0$ and thus $L - L_0 = (T - T_0)G^*$. Substituting this into $\Lambda - \Lambda_0 = G(L - L_0)$ yields the assertion. $\square$

It is well known that the operator $\Lambda - \Lambda_0$ is compact from $H^{-1/2}_0(\partial B)$ into $H^{1/2}_0(\partial B)$. This follows also from the factorization of Theorem 2.1 since $G$ is easily seen to be compact. The proof of the following theorem is quite standard and therefore omitted.

**Theorem 2.2** The operator $G : H^{-1/2}_0(\partial \Omega) \to H^{1/2}_0(\partial B)$ is compact and one-to-one with dense range.

The range of $G$ depends in a very specific way on the support $\Omega$ of $\gamma_1$. Later, in Theorem 2.6 we will construct functions $\varphi_y$, parametrized by points $y \in \mathbb{R}^n$, with the property that $y$ belongs to $\Omega$ if, and only if, $\varphi_y$ belongs to the range of $G$. Therefore, it is the aim to characterize the range of $G$ by properties which depend on the data $\Lambda - \Lambda_0$ only. Since $G$ and $\Lambda - \Lambda_0$ are related by (2.7) we first have to study the operators $T$ and $T_0$.

For real admittances $\gamma$ it is well known that $\Lambda$ depends monotonically on $\gamma$ in the sense that

$$z^*\left(\gamma(x) - \tilde{\gamma}(x)\right)z \geq 0 \quad \text{for all } z \in \mathbb{C}^n \quad \text{a.e. on } B$$

implies that $\langle \varphi, \Lambda \varphi \rangle \leq \langle \varphi, \tilde{\Lambda} \varphi \rangle$ for all $\varphi$. We prove a similar result for the operators $T$ and $T_0$ under one of the following **assumptions (A2a) or (A2b):**

There exists $c_2 > 0$ such that

(A2a) \[ z^*[\gamma_0(x) - \gamma(x)(\text{Re} \, \gamma(x))^{-1}\gamma(x)^*] z \geq c_2|z|^2 \]

for all $z \in \mathbb{C}^n$ and almost all $x \in \Omega$, or

(A2b) \[ \text{Re} [z^*\gamma_1(x)z] \geq c_2|z|^2 \]

for all $z \in \mathbb{C}^n$ and almost all $x \in \Omega$.

We note that for scalar functions $\gamma_0$ and $\gamma_1$ conditions (A2a) and (A2b) reduce to

$$\gamma_0(x) - \frac{|\gamma(x)|^2}{\text{Re} \, \gamma(x)} \geq c_2 \quad \text{or} \quad \text{Re} \, \gamma(x) - \gamma_0(x) \geq c_2,$$

respectively.

**Theorem 2.3** Let assumption (A1) hold. Then we have:

(a) $\text{Im} \langle (T - T_0)h, h \rangle \geq 0$ for all $h \in H^{1/2}_0(\partial \Omega)$.

(b) Assume that in addition (A2a) holds. Then there exists $c > 0$ with

$$\text{Re} \langle (T - T_0)h, h \rangle \geq c \|h\|^2_{H^{1/2}(\partial \Omega)} \quad \text{for all } h \in H^{1/2}_0(\partial \Omega).$$  (2.9a)
(c) Assume that (A2b) holds. Then there exists $c > 0$ with
\[
\text{Re} \langle (T_0 - T)h, h \rangle \geq c \|h\|^2_{H^{1/2}(\partial\Omega)} \quad \text{for all } h \in H^{1/2}(\partial\Omega).
\] (2.9b)

**Proof:** First, we compute since $Th = \partial_{\gamma_0} w_+ = \partial_{\gamma} w_-$:
\[
\langle Th, h \rangle = \int_{\partial\Omega} \partial_{\gamma_0} w_+ (\overline{w}_+ - \overline{w}_-) \, ds
\]
\[
= \int_{\partial B} \overline{w} \partial_{\gamma_0} w \, ds - \iint_{B \setminus \overline{\Omega}} \nabla^* \gamma_0 \nabla w \, dx - \iint_{\Omega} \nabla^* \gamma \nabla w \, dx
\]
\[
= - \iint_B \nabla^* \gamma \nabla w \, dx.
\]

Analogously,
\[
\langle T_0 h, h \rangle = - \iint_B \nabla^* \gamma_0 \nabla w_0 \, dx
\]
where $w_0$ solves (2.5) for $\gamma_0$ instead of $\gamma$. Therefore,
\[
\text{Im} \langle (T - T_0)h, h \rangle = -\text{Im} \iint_{\Omega} \nabla^* \gamma_1 \nabla w \, dx \geq 0
\]
by the assumptions on $\gamma$. Furthermore,
\[
\langle Th, h \rangle - \langle T_0 h, h \rangle = - \iint_B [\nabla^* \gamma \nabla w - \nabla^* \gamma_0 \nabla w_0] \, dx
\]
\[
= -2 \iint_B (\nabla w - \nabla w_0)^* \gamma \nabla w \, dx
\]
\[
+ \iint_B [\nabla^* \gamma_0 \nabla w_0 - 2 \nabla^* \gamma \nabla w + \nabla^* \gamma \nabla w] \, dx.
\]

Since $w - w_0 \in H^1_0(B)$ we note that by (2.6) for $\varphi = w - w_0$ the first integral vanishes, thus
\[
\langle (T - T_0)h, h \rangle = \iint_B [\nabla^* \gamma_0 \nabla w_0 - 2 \nabla^* \gamma \nabla w + \nabla^* \gamma \nabla w] \, dx.
\]

Interchanging the roles of $\gamma_0$ and $\gamma$ yields
\[
\langle (T_0 - T)h, h \rangle = \iint_B [\nabla^* \gamma \nabla w - 2 \nabla^* \gamma_0 \nabla w_0 + \nabla^* \gamma_0 \nabla w_0] \, dx.
\]
Let now (A2a) hold. Since Re γ is real, symmetric and positive definite there exists the positive square root \((\text{Re } \gamma)^{1/2}\). Therefore,

\[
\text{Re } \langle (T - T_0)h, h \rangle = \int_B \left[ \nabla w^* (\text{Re } \gamma) \nabla w - 2 \text{Re } \left( \nabla w^*_0 \gamma \nabla w \right) + \nabla w^*_0 \gamma - 0 \nabla w_0 \right] dx
\]

\[
= \int_B \left[ \left| (\text{Re } \gamma)^{1/2} \nabla w \right|^2 - 2 \text{Re } \left\{ \nabla w^*_0 \gamma (\text{Re } \gamma)^{-1/2} (\text{Re } \gamma)^{1/2} \nabla w \right\} + \nabla w^*_0 \gamma_0 \nabla w_0 \right] dx
\]

\[
= \int_B \left[ \left| (\text{Re } \gamma)^{1/2} \nabla w - (\text{Re } \gamma)^{-1/2} \gamma^* \nabla w_0 \right|^2 + \nabla w^*_0 \gamma_0 \nabla w_0 - \nabla w^*_0 \gamma (\text{Re } \gamma)^{-1} \gamma^* \nabla w^*_0 \right] dx
\]

\[
\geq \int_B \nabla w^*_0 \left[ \gamma_0 - \gamma (\text{Re } \gamma)^{-1} \gamma \right] \nabla w_0 dx
\]

\[
\geq c_2 \int_\Omega |\nabla w_0|^2 dx.
\]

The assertion now follows from standard arguments. Indeed, if \(\text{Re } \langle (T - T_0)h, h \rangle = 0\) then \(\nabla w_0 \equiv 0\) in \(\Omega\). Therefore, \(\partial_\gamma w_0^+ = \partial_\gamma w_0^- = 0\) on \(\partial \Omega\) and thus \(w_0 \equiv 0\) in \(B \setminus \overline{\Omega}\). This implies that \(h = w_0^+ - w_0^- = -w_0^-\) is constant on \(\partial \Omega\) and, therefore, has to vanish since \(h \in H^{1/2}_0(\partial \Omega)\). We have thus shown that

\[
\text{Re } \langle (T - T_0)h, h \rangle > 0 \text{ for all } h \in H^{1/2}_0(\partial \Omega), \ h \neq 0.
\]

Assume now that there exists a sequence \((h^{(j)})\) in \(H^{1/2}_0(\partial \Omega)\) with \(\|h^{(j)}\|_{H^{1/2}(\partial \Omega)} = 1\) and \(\text{Re } \langle (T - T_0)h^{(j)}, h^{(j)} \rangle \to 0, \ j \to \infty\). Then \(\|\nabla w^{(j)}\|_{L^2(\Omega)} \to 0\). We define the functions \(\tilde{w}^{(j)}\) in \(\Omega\) by

\[
\tilde{w}^{(j)} := w^{(j)} - \alpha_j \chi \quad \text{in } \Omega,
\]

where \(\alpha_j = \int_{\partial \Omega} w^{(j)}|_+ ds\) and \(\chi\) denotes the constant function with value 1/\(\int_{\partial \Omega} ds\). Since in \(\{w \in H^1(\Omega) : \int_{\partial \Omega} w ds = 0\}\) the norm \(w \mapsto \sqrt{\int_\Omega |\nabla w|^2 dx}\) is equivalent to the standard norm of \(H^1(\Omega)\) we conclude that \(\tilde{w}^{(j)} \to 0\) in \(H^1(\Omega)\). Therefore, also \(\partial_\gamma w^{(j)}|_+ = \partial_\gamma \tilde{w}^{(j)}|_+ \to 0\) in \(H^{-1/2}(\partial \Omega)\) and thus \(w^{(j)} \to 0\) in \(H^1(B \setminus \overline{\Omega})\) by the well-posedness of the Neumann problem in \(B \setminus \overline{\Omega}\). This yields that

\[
h^{(j)} + \alpha_j \chi = w^{(j)}|_+ - \tilde{w}^{(j)}|_- \to 0, \quad \text{in } H^{1/2}(\partial \Omega).
\]

Since also \(\int_{\partial \Omega} h^{(j)} ds = 0\) this implies \(\alpha_j \to 0\) and thus \(h^{(j)} \to 0\) in \(H^{1/2}(\partial \Omega)\). This contradicts the fact that \(\|h^{(j)}\|_{H^{1/2}(\partial \Omega)} = 1\). Therefore, (2.9a) is proven.
Let us now consider the case that (A2b) holds. Analogously as above we write

\[
\text{Re} \left\langle (T_0 - T) h, h \right\rangle = \iint_B \left[ |\gamma_0^{1/2} \nabla w_0|^2 - 2\text{Re} \left\{ (\gamma_0^{1/2} \nabla w_0)^* \gamma_0^{1/2} \nabla w \right\} + \text{Re} \left( \nabla w^* \gamma \nabla w \right) \right] \, dx
\]

\[
= \iint_B \left[ |\gamma_0^{1/2} (\nabla w_0 - \nabla w)|^2 + \text{Re} \left\{ \nabla w^* (\gamma - \gamma_0) \nabla w \right\} \right] \, dx
\]

\[
\geq c_2 \iint_{\Omega} |\nabla w|^2 \, dx.
\]

Now we can argue as before. \qed

2.3 An Excursion to Functional Analysis

We need the following facts from functional analysis. Let

\[
X \subset H \subset X^*
\]

be a Gelfand triple, i.e. \(H\) is a Hilbert space where \(H^*\) and \(H\) are identified, and \(X \subset H\) is a reflexive Banach space with dense imbedding. In our situation we think of the Gelfand triples \(H_0^{1/2}(\partial \Omega) \subset L_0^2(\partial \Omega) \subset H_0^{-1/2}(\partial \Omega)\) or \(H_0^{1/2}(\partial B) \subset L_0^2(\partial B) \subset H_0^{-1/2}(\partial B)\). By \(\langle \cdot, \cdot \rangle\) we denote both, the inner product in \(H\) as well as its extension to the dual system \(\langle X^*, X \rangle\).

Furthermore, for bounded operators \(A : X^* \to X\) we define the real and imaginary parts by

\[
\text{Re} \ A := \frac{1}{2} (A + A^*) \quad \text{and} \quad \text{Im} \ A := \frac{1}{2i} (A - A^*),
\]

respectively. Then \(\langle \varphi, A \varphi \rangle = \langle \varphi, (\text{Re} \ A) \varphi \rangle\) and \(\langle \varphi, A \varphi \rangle = \langle \varphi, (\text{Im} \ A) \varphi \rangle\).

If \(A\) is selfadjoint, i.e.

\[
\langle \varphi, A \psi \rangle = \overline{\langle \psi, A \varphi \rangle} \quad \text{for all } \varphi, \psi \in X^*,
\]

and the imbedding is compact then \(A|_H\) restricted to \(H\) is selfadjoint and compact and, therefore, admits a spectral decomposition in the form

\[
A \psi = \sum_{j=1}^{\infty} \lambda_j \langle \psi, \psi_j \rangle \psi_j, \quad \psi \in H,
\]

where \(\{ \lambda_j, \psi_j : j \in \mathbb{N} \}\) denotes an eigensystem of \(A|_H\). The operator \(|A|\) is then defined by

\[
|A| \psi = \sum_{j=1}^{\infty} |\lambda_j| \langle \psi, \psi_j \rangle \psi_j, \quad \psi \in H,
\]

If \(A\) is also non-negative, i.e.

\[
\langle \psi, A \psi \rangle \geq 0, \quad \psi \in X^*,
\]
then \( \lambda_j \geq 0 \) and the square root \( A^{1/2} \) is defined by

\[
A^{1/2} \psi = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \langle \psi, \psi_j \rangle \psi_j, \quad \psi \in H.
\]

From

\[
\| A^{1/2} \psi \|_H^2 = \langle A^{1/2} \psi, A^{1/2} \psi \rangle = \langle \psi, A \psi \rangle \leq \| \psi \|_{X^*} A \psi_\|_X \leq \| \psi \|_{X^*}^2 A \psi_\|_{X^*} \rightarrow x
\]

we observe that \( A^{1/2} \) has a bounded extension \( R \) to an operator from \( X^* \) into \( H \). Its adjoint \( R^* : H \rightarrow X \) coincides with \( A^{1/2} \) and shows that \( A^{1/2} \) is bounded from \( H \) into \( X \).

Therefore, we have shown that

\[
A = (A^{1/2}) (A^{1/2})^*.
\]

If \( A \) is even coercive, i.e.

\[
\langle \psi, A \psi \rangle \geq c \| \psi \|_{X^*}^2, \quad \psi \in X^*,
\]

for some \( c > 0 \) then \( A^{1/2} \) and \( (A^{1/2})^* \) are isomorphisms from \( H \) onto \( X \) and from \( X^* \) onto \( H \), respectively.

If, on the other hand, \( B : X \rightarrow X^* \) is selfadjoint and coercive, i.e.

\[
\langle Bx, x \rangle \geq c \| x \|_X^2, \quad \text{for all } x \in X,
\]

then we can apply the previous construction to \( A = B^{-1} \) which shows a decomposition in the form

\[
B = (B^{1/2})^* (B^{1/2})
\]

where \( B^{1/2} \) and \( (B^{1/2})^* \) are isomorphisms from \( X \) onto \( H \) and from \( H \) onto \( X^* \), respectively.

We also need the following result from functional analysis:

**Lemma 2.4** Let \( H_1 \) and \( H_2 \) be Hilbert spaces, \( X \) be a reflexive Banach space, \( A_j : H_j \rightarrow X \) compact, bounded and one-to-one with \( A_1 A_1^* = A_2 A_2^* \). Then the ranges of \( A_1 \) and \( A_2 \) coincide.

**Proof:** Let \( A \) be \( A_1 \) or \( A_2 \). We recall the following result (cf. [15]):

\[
\varphi \in \mathcal{R}(A) \quad \text{if and only if} \quad \inf \{ \langle A A^* \psi, \psi \rangle : \psi \in X^*, \langle \psi, \varphi \rangle = 1 \} > 0.
\]

This proves the theorem since the inf-conditions coincide for \( A = A_1 \) and \( A = A_2 \) by assumption. \( \square \)

### 2.4 The Main Result

We apply the results of the previous subsection to the factorization (2.7). We define \( B : = \text{Re} T - T_0 \) if (A2a) holds and \( B : = T_0 - \text{Re} T \) if (A2b) holds. Then

\[
|\text{Re} \Lambda - \Lambda_0| = GBG^*
\]
and $B : H^{1/2}_0(\partial \Omega) \to H^{-1/2}_0(\partial \Omega)$ is selfadjoint and coercive. Taking the square roots as in the previous subsections of $A = |\Re \Lambda - \Lambda_0|$ and $B$ yields

$$
|\Re \Lambda - \Lambda_0|^{1/2}(|\Re \Lambda - \Lambda_0|^{1/2})^* = |\Re \Lambda - \Lambda_0| = (B^{1/2})^* B^{1/2} G^* = [G(B^{1/2})^*] [G(B^{1/2})]^*.
$$

Application of Lemma 2.4 yields that the ranges of $|\Re \Lambda - \Lambda_0|^{1/2}$ and $G(B^{1/2})^*$ coincide. Since $(B^{1/2})^*$ is an isomorphism we have shown:

**Theorem 2.5** Let assumptions (A1) and (A2a) or (A2b) hold. Then the ranges of the operators $|\Re \Lambda - \Lambda_0|^{1/2} : L^2_0(\partial B) \to H^{1/2}_0(\partial B)$ and $G : H^{-1/2}_0(\partial \Omega) \to H^{1/2}_0(\partial B)$ coincide. We can reformulate this as follows:

Let $\{\lambda_j, \psi_j : j \in \mathbb{N}\}$ be an eigensystem of $\Re \Lambda - \Lambda_0$. Then $\varphi \in H^{1/2}_0(\partial B)$ belongs to the range of $G$ if, and only if,

$$
\sum_{j=1}^{\infty} \frac{|\langle \varphi, \psi_j \rangle|^2}{|\lambda_j|} < \infty.
$$

**Proof:** As seen by (2.10) for $A = |\Re \Lambda - \Lambda_0|$ and the theorem of Picard (see [12]), $\varphi$ belongs to the range of $|\Re \Lambda - \Lambda_0|^{1/2}$ if and only if $\sum_{j=1}^{\infty} |\langle \varphi, \psi_j \rangle|^2 / |\lambda_j| < \infty$. \qed

This theorem solves our first problem to characterize the range of $G$ by our data $\Lambda - \Lambda_0$. As mentioned before, the second task is to construct singular solutions of (2.1) by using the fundamental solution $\Phi$ of $\Div (\gamma_0 \nabla u) = 0$. We follows exactly the arguments of [1]. It is here where we have to assume that $\gamma_0 \in C^{2,\alpha}(B)$ to assure existence of the fundamental solution $\Phi$ of (2.2), see, e.g., [18], Section III.19. The fundamental solution has the following properties:

(i) For any $f \in C^{0,\alpha}(B)$ the function $u(x) = -\int_B f(y) \Phi(y) \, dy$, $x \in B$, solves the differential equation $\Div (\gamma_0 \nabla u) = f$ in $B$,

(ii) $\int_{\partial B} \partial \gamma_0 \Phi(\cdot, y) \, ds = 1$ for every $y \in B$, and

(iii) $\Phi(x, y)$ has a singularity at $x = y$ of the following type: There exists $c > 0$ and $\alpha \in (0, 1)$ with

$$
|\Phi(x, y) - L(x, y)| \leq c |x - y|^{\alpha + 2 - n},
$$

$$
\left| \frac{\partial [\Phi(x, y) - L(x, y)]}{\partial x_i} \right| \leq c |x - y|^{\alpha + 1 - n},
$$

$$
\left| \frac{\partial^2 [\Phi(x, y) - L(x, y)]}{\partial x_i \partial x_j} \right| \leq c |x - y|^{\alpha - n},
$$

for $x, y \in B$, $x \neq y$, $i, j = 1, \ldots, n$, where the Levi function $L$ is defined by

$$
L(x, y) := \begin{cases} 
\frac{1}{4\pi \sqrt{\det \gamma_0(y)}} \frac{1}{\sqrt{(x - y)^\top \gamma_0(y)^{-1} (x - y)}}, & n = 3, \\
-\frac{1}{4\pi \sqrt{\det \gamma_0(y)}} \ln [(x - y)^\top \gamma_0(y)^{-1} (x - y)], & n = 2.
\end{cases}
$$
We construct the Neumann function as follows: First, let $\chi \in C(\partial B)$ be any function with $\int_{\partial B} \chi \, ds = 1$. For fixed $y \in B$ let $u = u_y \in H^1_0(B)$ be the unique solution of (2.1) for $f = \partial_{\gamma_0} \Phi(\cdot, y) - \chi$ on $\partial B$. Note that $f \in H^{-1/2}(\partial B)$ by (ii) and the choice of $\chi$. Then $N(x, y) = \Phi(x, y) - u_y(x)$ is the Neumann function, i.e., satisfies $\text{div} \, \left( \gamma_0 \nabla N(\cdot, y) \right) = 0$ in $B \setminus \{y\}$ and $\partial_{\gamma_0} N(\cdot, y) = \chi$ on $\partial B$. We fix some unit vector $\hat{a} \in \mathbb{R}^n$ and define the function $\varphi_y \in H^{1/2}(\partial B)$ by

$$
\varphi_y(x) = \hat{a}^T \nabla_y N(x, y) = \hat{a}^T \nabla_y \left[ \Phi(x, y) - u_y(x) \right], \quad x \in \partial B. \tag{2.11}
$$

Then we can show:

**Theorem 2.6** Let $\gamma_0 \in C^{2,\alpha}(B)$ for some $\alpha \in (0, 1]$ and $\varphi_y$ be defined by (2.11) for $y \in B$. Then $\varphi_y$ belongs to the range of $G$ if and only if $y \in \Omega$.

**Proof:** First, let $y \in \Omega$. Then, obviously, $G \psi = \varphi_y$ for $\psi = \partial_{\gamma_0} \left[ \hat{a}^T \nabla_y N(\cdot, y) \right]$ on $\partial B$.

Second, let $y \notin \Omega$ and assume, on the contrary, that $G \psi = \varphi_y$ for some $\psi \in H^{-1/2}(\partial \Omega)$. By $v \in H^1(B \setminus \bar{\Omega})$ we denote the corresponding solution of (2.4). Set $u = \hat{a}^T \nabla_y N(\cdot, y)$ in $B \setminus \bar{\Omega}$. From $\partial_{\gamma_0} v = 0 = \partial_{\gamma_0} u$ and $v = \varphi_y = u$ on $\partial B$ we conclude that $u$ and $v$ coincide on $B \setminus \{\bar{\Omega} \cup \{y\}\}$ since uniqueness holds for the Cauchy problem (Miranda [18]). Since also $u \in H^1(\Omega)$ we conclude that $u \in H^1_0(B)$ and thus $u \equiv 0$ by the uniqueness of the Neumann problem. Now we use the asymptotic behaviour of $u$ in the form

$$
u(x) = - \frac{\hat{a}^T \gamma_0(y)^{-1}(y - x)}{2(n - 1)\pi \sqrt{\det \gamma_0(y) \left[ (x - y)^T \gamma_0(y)^{-1}(x - y) \right]^{n/2}}} + w(x)
$$

where

$$
\left| w(x) \right| \leq \left\{ \begin{array}{ll}
c|\ln |x - y||, & n = 2, \\
c/|x - y|, & n = 3.
\end{array} \right.
$$

Since $w \in H^1(B)$ this singular behaviour contradicts the fact that $u$ vanishes. \hfill \Box

The combination of Theorems 2.5 and 2.6 yields the main result of this section:

**Theorem 2.7** Let assumptions (A1) and (A2a) or (A2b) hold. Let $\gamma_0 \in C^{2,\alpha}(B)$ for some $\alpha \in (0, 1]$. Then $y \in \Omega$ if, and only if,

$$
\sum_{j=1}^{\infty} \frac{\left| \langle \varphi_y, \psi_j \rangle \right|^2}{|\lambda_j|} < \infty,
$$

where $\varphi_y$ is defined by (2.11) for $y \in \mathbb{R}^n$ and $\{\lambda_j, \psi_j : j \in \mathbb{N}\}$ denotes an eigensystem of the selfadjoint operator $\text{Re} \Lambda - \Lambda_0$.

Therefore, defining the function $W : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$
W(y) := \left( \sum_{j=1}^{\infty} \frac{\left| \langle \varphi_y, \psi_j \rangle \right|^2}{|\lambda_j|} \right)^{-1}, \quad y \in \mathbb{R}^n,
$$

we observe that $W(y) > 0$ if, and only if, $y \in \Omega$. Therefore, the signum of $W$ is just the characteristic function of $\Omega$. Contour plots of $W$ for several inverse scattering problems are shown in, e.g., [13, 10, 17].
3 Extension to Non-Coercive Equations

Let $B$, $\Omega$, $\gamma_0$ and $\gamma_1$ as before in Section 2. We recall the assumptions (A1) and (A2) for the convenience of the reader but will restrict ourselves to (A2a). The case (A2b) is treated analogously. We refer to (A3) for the following assumption which combines (A1) and (A2a):

(A3) Let $\gamma_0 \in C^{2,\alpha}(\overline{B})$ be real, symmetric and uniformly positive definite, i.e. there exists $c_0 > 0$ with

$$z^* \gamma_0(x) z \geq c_0 |z|^2 \text{ for all } z \in \mathbb{C}^n \text{ and all } x \in \overline{B}. \quad (3.12)$$

We assume that $\gamma$ has the form

$$\gamma(x) = \begin{cases} 
\gamma_0(x), & x \in B \setminus \Omega, \\
\gamma_0(x) + \gamma_1(x), & x \in \Omega,
\end{cases}$$

where $\gamma_1 \in L^\infty(\Omega)$ is complex valued and symmetric a.e. on $\Omega$. Here, $\Omega \subset \mathbb{R}^n$ denotes a domain with $C^2$-boundary $\partial \Omega$ such that $\overline{\Omega} \subset B$ and $B \setminus \overline{\Omega}$ is connected. We assume, furthermore, that there exists $c_1 > 0$ with

$$\text{Re} [z^* \gamma(x) z] \geq c_1 |z|^2 \text{ and } \text{Im} [z^* \gamma_1(x) z] \leq 0 \text{ for all } z \in \mathbb{C}^n \quad (3.13)$$

and almost all $x \in \Omega$. Furthermore, there exists $c_2 > 0$ such that

$$z^* [\gamma_0(x) - \gamma(x) \left(\text{Re} \gamma(x)\right)^{-1} \gamma(x)^*] z \geq c_2 |z|^2 \text{ for all } z \in \mathbb{C}^n \text{ and almost all } x \in \Omega. \quad (3.14)$$

Finally, let $q_0, q_1 \in L^\infty(B)$ be scalar functions, $q_0$ real valued, with $\text{Im} q_1 \geq 0$ ans $\text{supp} q_1 \subset \Omega$.

We consider the boundary value problem which replaces (2.1):

$$\text{div} (\gamma \nabla u) + qu = 0 \text{ in } B, \quad \partial_\gamma u = f \text{ on } \partial B, \quad (3.15)$$

where $q = q_0 + q_1$ on $B$. The corresponding unperturbed situation is described by

$$\text{div} (\gamma_0 \nabla u_0) + q_0 u_0 = 0 \text{ in } B, \quad \partial_{\gamma_0} u = f \text{ on } \partial B. \quad (3.16)$$

We make a second assumption on the solvability of the boundary value problem:

(A4) Let (3.15) and (3.16) be well posed, i.e. for any $f \in H^{-1/2}(\partial B)$ there exist unique solutions $u, u_0 \in H^1(B)$, and the solutions depend continuously on $f$.

As it is well known (see [18]), this assumption is, e.g., satisfied if also $q$ is real valued and $q(x) < 0$ and $q_0(x) < 0$ on $B$.

Under assumption (A4) the Neumann-Dirichlet operators $\Lambda, \Lambda_0 : H^{-1/2}(\partial B) \to H^{1/2}(\partial B)$, given by $\Lambda f = u|_{\partial B}$ and $\Lambda_0 f = u_0|_{\partial B}$ are well defined. For technical reasons we have to make the following assumptions:

(A5) The Neumann boundary value problem

$$\text{div} (\gamma_0 \nabla v) + q_0 v = 0 \text{ in } B \setminus \overline{\Omega}, \quad \partial_{\gamma_0} v = \psi \text{ on } \partial \Omega, \quad \partial_\gamma v = 0 \text{ on } \partial B, \quad (3.17)$$
is well posed for \( \psi \in H^{-1/2}(\partial \Omega) \).

Then the operator \( G : H^{-1/2}(\partial \Omega) \rightarrow H^{1/2}(\partial B) \), \( \psi \mapsto v|_{\partial B} \), where \( v \) solves (3.17) is well defined and bounded. We omit the proof of the following theorem since it is only a slight modification of that of Theorem 2.1.

**Theorem 3.1** The following factorization holds:

\[
\Lambda - \Lambda_0 = G (T - T_0) G^*.
\]

Here, \( T : H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega) \) is given by \( Th = \partial_{\gamma_0} w_+ \) on \( \partial \Omega \) where \( w \in H^1(B \setminus \overline{\Omega}) \cap H^1(\Omega) \) solves the boundary value problem

\[
\text{div} (\gamma \nabla w) + qw = 0 \quad \text{in } B \setminus \partial \Omega, \quad \partial_{\gamma_0} w = 0 \quad \text{on } \partial B, \quad (3.19a)
\]

\[
w_+ - w_- = h \quad \text{on } \partial \Omega, \quad \partial_{\gamma_0} w_+ - \partial_{\gamma} w_- = 0 \quad \text{on } \partial \Omega. \quad (3.19b)
\]

Again, the operator \( T_0 \) is defined analogously with \( \gamma \) and \( q \) replaced by \( \gamma_0 \) and \( q_0 \), respectively. Then \( T_0 \) is selfadjoint and the adjoint \( T^* \) of \( T \) is given by the trace of the boundary value problem where \( \gamma \) and \( q \) are replaced by their complex conjugates. The operator \( G \) is again one-to-one and compact with dense range. Also, (3.18) yields

\[
\text{Re } \Lambda - \Lambda_0 = G (\text{Re } T - T_0) G^*.
\]

However, opposed to the situation of the previous section the selfadjoint operator \( \text{Re } T - T_0 \) fails to be coercive but is a compact perturbation of a coercive operator. This is shown in the following lemma.

**Lemma 3.2** Let \( \hat{T}, \hat{T}_0 : H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}_0(\partial \Omega) \) be the operators \( T, T_0 \) for the special case \( q \equiv q_0 \equiv -1 \). Then the following holds:

(a) \( \text{Re } \hat{T} - \hat{T}_0 \) is coercive in the sense of (2.9a), i.e.

\[
\text{Re } \langle (\hat{T} - \hat{T}_0) h, h \rangle \geq c \| h \|_{H^{1/2}(\partial \Omega)}^2 \quad \text{for all } h \in H^{1/2}(\partial \Omega)
\]

for some \( c > 0 \).

(b) \( T - \hat{T} \) and \( T_0 - \hat{T}_0 \) are bounded from \( H^{1/2}(\partial \Omega) \) into itself and thus compact from \( H^{1/2}(\partial \Omega) \) into \( H^{-1/2}(\partial \Omega) \).

**Proof:** (a) This part is proven exactly as the corresponding part of Theorem 2.3. One derives first a representation of the form

\[
\langle (\hat{T} - \hat{T}_0) h, h \rangle = \iint_B \left[ \nabla w_0^* \gamma_0 \nabla w_0 - 2 \nabla w_0^* \gamma \nabla w + \nabla w^* \gamma \nabla w \right] dx
\]

\[
+ \iint_B \left( |w_0|^2 - 2 \overline{w_0} w + |w|^2 \right) dx
\]

and concludes as in the proof of Theorem 2.3 by noting that

\[
\text{Re } \iint_B \left( |w_0|^2 - 2 \overline{w_0} w + |w|^2 \right) dx = \iint_B |w_0 - w|^2 dx \geq 0.
\]
(b) We consider only the operator $T - \hat{T}$ and note that $(T - \hat{T})h = \partial_{\gamma_0} \hat{w}$ on $\partial\Omega$ where $\hat{w} \in H^1(B)$ solves
\[
\text{div} (\gamma \nabla \hat{w}) = -(1 + q) w \quad \text{in } B, \quad \partial_{\gamma_0} \hat{w} = 0 \quad \text{on } \partial B.
\] (3.20)

The dependence on $h$ is given implicitly through $w$. The solution operator $h \mapsto (w|_{\Omega}, w|_{B\setminus\Omega})$ is bounded from $H^{1/2}(\partial\Omega)$ into $H^1(\Omega) \times H^1(B\setminus\Omega)$, thus $h \mapsto qw$ is bounded from $H^{1/2}(\partial\Omega)$ into $L^2(B)$. Standard regularity results for the boundary value problem (3.20) yields the existence of $c > 0$ with $\|\hat{w}\|_{H^2(B)} \leq c \|qw\|_{L^2(B)}$. Therefore, $h \mapsto \hat{w}$ is bounded from $H^{1/2}(\partial\Omega)$ into $H^2(B)$ which yields that $T - \hat{T}$ is bounded from $H^{1/2}(\partial\Omega)$ into itself. \hfill $\square$

It is the aim to apply the following result from functional analysis which is a slight modification of Theorem 4.4 in [16] (see also [9]. We will give a proof in the Appendix for the convenience of the reader.

**Theorem 3.3** Let $X \subset U \subset X^*$ and $Y \subset H \subset Y^*$ be Gelfand triples with Hilbert spaces $U$ and $H$ and reflexive Banach spaces $X$ and $Y$ such that the imbeddings are dense. Furthermore, let $F : Y^* \to Y$, $G : X^* \to Y$, and $D : X \to X^*$ be linear and bounded operators with
\[
F = GDG^*.
\] (3.21)

We make the following assumptions:

(a) $G$ is one-to-one and compact with dense range in $Y$.

(b) $\text{Re } D$ has the form $\text{Re } D = C + K$ with some coercive operator $C : X \to X^*$ and compact operator $K$.

(c) $\text{Im } D$ is non-negative on $X$, i.e. $\text{Im } \langle D\varphi, \varphi \rangle \geq 0$ for all $\varphi \in X$.

(d) $\text{Re } D$ is one-to-one (and thus an isomorphism onto $X$) or $\text{Im } D$ is positive on $X$, i.e. $\text{Im } \langle D\varphi, \varphi \rangle > 0$ for all $\varphi \in X$, $\varphi \neq 0$.

Then the operator $F^\# := |\text{Re } F| + \text{Im } F$ is also positive, and the ranges of $G : X^* \to Y$ and $F^\# : H \to Y$ coincide.

In order to apply this result to $X = H^{1/2}(\partial\Omega)$, $Y = H^{1/2}(\partial B)$, $F = \Lambda - \Lambda_0$, and $D = T - T_0$ we have to check assumptions (b), (c), and (d). Lemma 3.2 allows us to write $\text{Re } T - T_0$ in the form
\[
\text{Re } T - T_0 = C + K
\] (3.22)
where $C = \text{Re } \hat{T} - \hat{T}_0$ is coercive on $H^{1/2}(\partial\Omega)$ and
\[
K = \text{Re } T - T_0 - C = \text{Re } (T - \hat{T}) - (T_0 - \hat{T}_0)
\]
is compact from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$. This proves assumption (b) of Theorem 3.3.
For (c) we have to show that \( \text{Im} \langle (T - T_0) h, h \rangle \geq 0 \) on \( H^{1/2}(\partial \Omega) \). Indeed, as in the proof of Theorem 2.3 we have

\[
\langle T h, h \rangle = \int_{\partial \Omega} \partial_{\gamma_0} w_+ (\overline{w}_+ - \overline{w}_-) \, ds
\]

\[
= \int_{\partial B} \overline{w} \partial_{\gamma_0} w \, ds + \iint_{B \setminus \Omega} \left[ q_0 |w|^2 - \nabla w^* \gamma_0 \nabla w \right] \, dx + \iint_{\Omega} \left[ q |w|^2 - \nabla w^* \gamma \nabla w \right] \, dx
\]

\[
= \iint_{B} \left[ q |w|^2 - \nabla w^* \gamma \nabla w \right] \, dx
\]

and thus

\[
\text{Im} \langle (T - T_0) h, h \rangle = \text{Im} \langle T h, h \rangle = \iint_{\Omega} \left[ \text{Im} q_1 |w|^2 - \text{Im} (\nabla w^* \gamma_1 \nabla w) \right] \, dx \geq 0 \quad (3.23)
\]

by our assumptions on \( q_1 \) and \( \gamma_1 \).

It remains to check assumption (d). The following lemma characterizes the null space \( \mathcal{N}(T - T_0) \) of \( T - T_0 \) by solution of the following homogeneous “interior transmission problem”:

\[
\begin{align*}
\text{div} (\gamma \nabla w) + qw &= 0 \quad \text{in} \ \Omega, \\
\text{div} (\gamma_0 \nabla w_0) + q_0 w_0 &= 0 \quad \text{in} \ \Omega, \\
\partial_\gamma w_- &= \partial_{\gamma_0} w_0 - \text{on} \ \partial \Omega, \\
w_- &= w_0 - \text{on} \ \partial \Omega.
\end{align*}
\]

(3.24)

In general, this coupled boundary value problem is understood in the variational sense, i.e. \( w, w_0 \in L^2(\Omega) \) with \( w - w_0 \in H^2_0(\Omega) \).

**Lemma 3.4** (a) Let \( h \in H^{1/2}(\partial \Omega) \) with \( (T - T_0) h = 0 \) on \( \partial \Omega \) and let \( w \) and \( w_0 \) be the corresponding solutions of (3.19) for \( \gamma, q \) and \( \gamma_0, q_0 \), respectively. Then the restriction \( (w|_{\Omega}, w_0|_{\Omega}) \) solves (3.24).

(b) Let \( (w, w_0) \) be a solution of (3.24) with \( w, w_0 \in H^1(\Omega) \). Define \( \hat{w} \in H^1(B \setminus \overline{\Omega}) \) by the solution of the Neumann problem

\[
\text{div} (\gamma_0 \nabla \hat{w}) + q_0 \hat{w} = 0 \quad \text{in} \ B \setminus \overline{\Omega}, \quad \partial_{\gamma_0} \hat{w}_+ = \partial_{\gamma} w_- \quad \text{on} \ \partial \Omega, \quad \partial_{\gamma_0} \hat{w} = 0 \quad \text{on} \ \partial B.
\]

Then \( h := \hat{w}_+ - w_- \in \mathcal{N}(T - T_0) \).

**Proof:** (a) \( w \) and \( w_0 \) satisfy

\[
\begin{align*}
\text{div} (\gamma \nabla w) + qw &= 0 \quad \text{in} \ B \setminus \partial \Omega, \\
\text{div} (\gamma_0 \nabla w_0) + q_0 w_0 &= 0 \quad \text{in} \ B \setminus \partial \Omega, \\
\partial_{\gamma_0} w &= \partial_{\gamma_0} w_0 = 0 \quad \text{on} \ \partial B, \\
\partial_{\gamma_0} w_+ &= \partial_{\gamma} w_- = \partial_{\gamma_0} w_0 - \partial_{\gamma_0} w_+ = \partial_{\gamma_0} w_+ \quad \text{on} \ \partial \Omega, \\
w_+ - w_- &= h = w_0+ - w_0- \quad \text{on} \ \partial \Omega.
\end{align*}
\]

By the uniqueness of the Neumann problem in \( B \setminus \overline{\Omega} \) we conclude that \( w \) and \( w_0 \) coincide in \( B \setminus \overline{\Omega} \). Therefore, \( w, w_0 \in H^1(\Omega) \) satisfy (3.24).
Let \( h = h_0 \) and thus \( (T - T_0)h = \partial_{\gamma_0}(w_+ - w_+) = 0 \) on \( \partial \Omega \). ∎

Since Re \( T - T_0 = T - T_0 \) for real \( \gamma \) the first part of the following theorem has thus been proven.

**Theorem 3.5** Let assumptions (A3), (A4) and (A5) hold.

(a) Let \( \gamma_1 \) and \( q_1 \) be real valued and the interior transmission problem (3.24a), (3.24b) admit only the trivial solution \( w \equiv w_0 \equiv 0 \) in \( \Omega \).

Then the ranges of the operators \( |\Lambda - \Lambda_0|^{1/2} : L^2(\partial B) \rightarrow H^{1/2}(\partial B) \) and \( G : H^{-1/2}(\partial \Omega) \rightarrow H^{1/2}(\partial B) \) coincide.

(b) Let \( \gamma_1 \in C^{1,\alpha}(\Omega) \) for some \( \alpha \in (0, 1] \) and let there exist an open set \( U \subset \Omega \) with

(i) \( q(x) \neq 0 \) and Im \( z^{*}\gamma_1(x)z \) < 0 for all \( z \in \mathbb{R}^n \), \( z \neq 0 \), and almost all \( x \in U \), or

(ii) \( \text{Im } q_1 > 0 \) almost everywhere on \( U \).

Then the ranges of the operators \( [|\text{Re } \Lambda - \Lambda_0| + \text{Im } \Lambda]^{1/2} : L^2(\partial B) \rightarrow H^{1/2}(\partial B) \) and \( G : H^{-1/2}(\partial \Omega) \rightarrow H^{1/2}(\partial B) \) coincide.

Let \( \{\lambda_j, \psi_j : j \in \mathbb{N}\} \) denote an eigensystem of the selfadjoint and positive operator \( |\Lambda - \Lambda_0| \) or \( |\text{Re } \Lambda - \Lambda_0| + \text{Im } \Lambda \) of cases (a) or (b), respectively. Then \( \varphi \in H^{1/2}(\partial B) \) belongs to the range of \( G \) if, and only if,

\[
\sum_{j=1}^{\infty} \frac{|\langle \varphi, \psi_j \rangle|^2}{\lambda_j} < \infty.
\]

**Proof:** It remains to study the case (b). To check assumption (d) of Theorem 3.3 we have to show that Im \( D = \text{Im } T \) is positive on \( H^{1/2}(\partial \Omega) \). From (3.23) we conclude that Im \( \langle (T - T_0)h, h \rangle = 0 \) implies that \( w \) vanishes on \( U \). Indeed, this follows immediately under assumption (ii). If (i) holds then from (3.23) we conclude first that \( w \) is constant in \( U \). The differential equation yields \( qw \equiv 0 \) in \( U \) which then also implies that \( w \) vanishes on \( U \). Now the unique continuation property holds under our assumptions on \( \gamma \) (see [18]). Therefore, \( w \) vanishes in \( \Omega \). The uniqueness of the Neumann problem in \( B \setminus \Omega \) yields \( w = 0 \) in \( B \setminus \Omega \) which proves that also \( h = w_+ - w_- = 0 \) on \( \partial \Omega \). ∎

Now we can proceed exactly as in the previous section. We introduce the Neumann function \( N \) of the differential equation div \((\gamma_0\nabla u) + q_0 u = 0\) in \( B \) and define, for any \( y \in B \), the function \( \varphi_y \in H^{1/2}(\partial B) \) by

\[
\varphi_y(x) = \hat{a}^T \nabla_y N(x, y) = \hat{a}^T \nabla_y [\Phi(x, y) - u_y(x)], \quad x \in \partial B,
\]

for some fixed unit vector \( \hat{a} \in \mathbb{R}^n \). Then Theorem 2.6 holds literally and we arrive at the main result of this section:
**Theorem 3.6** Let assumptions (A3), (A4) and (A5) hold. Furthermore, let assumptions (a) or (b) of Theorem 3.5 hold. Then \( y \in \Omega \) if, and only if,

\[
\sum_{j=1}^{\infty} \frac{|\langle \varphi_y, \psi_j \rangle|^2}{\lambda_j} < \infty ,
\]

where \( \varphi_y \) is defined by (3.25) for \( y \in B \) and \( \{ \lambda_j, \psi_j : j \in \mathbb{N} \} \) denotes an eigensystem of the selfadjoint and positive operator \(|\text{Re} \Lambda - \Lambda_0| + \text{Im} \Lambda|\).

Formula (3.26) provides an explizit representation of the characteristic function \( \chi_{\Omega}(y) \) by the Neumann-Dirichlet operator. Indeed, we have

\[
\chi_{\Omega}(y) = \text{sign} \left[ \sum_{j=1}^{\infty} \frac{|\langle \varphi_y, \psi_j \rangle|^2}{\lambda_j} \right]^{-1} , \quad y \in B.
\]

To the knowledge of the author, the transmission problem (3.24) and, in particular, its non-homogeneous counterpart has not been studied before in this generality. Up to regularity, the solution space of (3.24) is isomorphic to the null space of \( T - T_0 \) which is finite dimensional by Lemma 3.2. For scattering problems by inhomogeneous media with \( \gamma \equiv \gamma_0 \equiv 1 \) and \( q_0 = k^2, \quad q = k^2 \) with \( k \in \mathbb{R} \), it has been shown in [19] or [7], Section 8.6 that, under certain technical assumptions on \( n \), there exists at most a discrete set of “eigenvalues” \( k^2 \) which can only accumulate at infinity.

**Appendix**

**Proof of Theorem 3.3:**

First, we show that we can assume without loss of generality that \( X = U \) and \( Y = H \) are Hilbert spaces and \( C \) is the identity in \( U \). Indeed, if we restrict \( F \) to \( H \) and consider \( G \) as an operator from \( X^* \) to \( H \) then the factorization (3.18) holds as well as the assumptions (a)–(d). Also, the ranges of \( F \) and \( F_{\#}^{1/2} \) do not depend on the topology of the range space.

Therefore, we can assume that \( Y = H \). Since \( C \) is coercive there exists a coercive “square root” \( C^{1/2} : U \to X^* \) with \( C = (C^{1/2})^* C^{1/2} \), see Subsection 2.3. We denote its inverse by \( C^{-1/2} \). Then

\[
F = \begin{bmatrix} G (C^{1/2})^* \end{bmatrix} \begin{bmatrix} (C^{-1/2})^* DC^{-1/2} \end{bmatrix} \begin{bmatrix} G (C^{1/2})^* \end{bmatrix}^* = \tilde{G} \tilde{D} \tilde{G}^* \quad \text{and}
\]

\[
\text{Re} F = \begin{bmatrix} G (C^{1/2})^* \end{bmatrix} \begin{bmatrix} I + (C^{-1/2})^* KC^{-1/2} \end{bmatrix} \begin{bmatrix} G (C^{1/2})^* \end{bmatrix}^* = \tilde{G} \begin{bmatrix} I + \tilde{K} \end{bmatrix} \tilde{G}^* \]

and \( \tilde{G} = G (C^{1/2})^* : U \to H \) and \( \tilde{D} = (C^{-1/2})^* DC^{-1/2} : U \to U \) satisfy the assumptions (a)–(d). Therefore, we can also assume that \( X = U \) is a Hilbert space.

We note from (3.18) that \( F \) and therefore also \( \text{Re} F \) and \( \text{Im} F \) are compact. Let \( \{(\lambda_j, \psi_j) : j \in \mathbb{N}\} \) of \( \text{Re} F \) be a (complete) orthonormal eigensystem and set

\[
H^+ = \text{span} \{ \psi_j : \lambda_j > 0 \} \quad \text{and} \quad H^- = \text{span} \{ \psi_j : \lambda_j \leq 0 \},
\]

and note that \( H = H^+ \oplus H^- \) is orthogonal, and \( \text{Re} F \) keeps the spaces \( H^\pm \) invariant. If \( P^\pm : H \to H^\pm \) denotes the orthogonal projection then

\[
|\text{Re} F| = (P^+ - P^-)(\text{Re} F) = (\text{Re} F)(P^+ - P^-) = G(\text{Re} D) G^*(P^+ - P^-). \quad (3.27)
\]
In the first part of the proof we will derive a factorization of $F_\#$ in the form $F_\# = GD_\#G^\ast$. This is achieved by constructing bounded operators $Q^\pm$ in $U$ with
\[
(\text{Re} \, D) \ G^\ast \ P^\pm = (\text{Re} \, D) \ Q^\pm \ G^\ast.
\]
(3.28)
since then with (3.27)
\[
F_\# = G \ (\text{Re} \, D) \ G^\ast \ (P^+ - P^-) + G \ (\text{Im} \, D) \ G^\ast = G \ ((\text{Re} \, D) \ (Q^+ - Q^-) + \text{Im} \, D) \ G^\ast.
\]
(3.29)

In the second part we show that $D_\#$ is coercive and proceed exactly as in Theorem 2.5. We note that $G^*(H^+) + G^*(H^-)$ is dense in $U$. First we show that the subspace $U^- := G^*(H^-)$ of $U$ is finite dimensional. Let $\{(\nu_j, \varphi_j) : j \in \mathbb{N}\}$ be an eigensystem of the selfadjoint and compact operator $K$ and define
\[
V^+ := \text{span} \{\varphi_j : 1 + \nu_j > 0\}, \quad V^- := \text{span} \{\varphi_j : 1 + \nu_j \leq 0\}.
\]
Then $V^-$ is finite dimensional since $\nu_j \to 0$. Furthermore, $U = V^+ \oplus V^-$, the sum is orthogonal, and there exists $c > 0$ with
\[
\langle (\text{Re} \, D) \varphi, \varphi \rangle \geq c \|\varphi\|^2 \quad \text{for all} \quad \varphi \in V^+.
\]

Every $\varphi = G^* \psi^- \in U^-$, i.e. $\psi^- \in H^-$, has a unique splitting in the form $\varphi = \varphi^+ + \varphi^-$ with $\varphi^\pm \in V^\pm$ and thus
\[
0 \geq \langle (\text{Re} \, F) \psi^-, \psi^- \rangle = \langle (\text{Re} \, D) G^* \psi^-, G^* \psi^- \rangle = \langle (\text{Re} \, D) (\varphi^+ + \varphi^-), \varphi^+ + \varphi^- \rangle
\]
\[
= \langle (\text{Re} \, D) \varphi^+, \varphi^+ \rangle + \langle (\text{Re} \, D) \varphi^-, \varphi^- \rangle
\]
\[
\geq c \|\varphi^+\|^2 - \|D\| \|\varphi^-\|^2 = c \|\varphi\|^2 - (\|D\| + c) \|\varphi^-\|^2.
\]

If we denote the orthogonal projector from $U$ onto $V^-$ by $R^-$ then $\varphi^- = R^- \varphi$ and thus
\[
\|\varphi\| \leq \sqrt{1 + \frac{\|D\|}{c}} \|R^- \varphi\| \quad \text{for all} \quad \varphi \in U^-.
\]
This shows that the operator
\[
R^- \ |_{U^-} : U^- \to V^-
\]
is one-to-one. Therefore, also $U^-$ is finite dimensional. From a standard result from functional analysis we conclude that $U^- + G^*(H^+)$ is closed in $U$. Since the sum is also dense in $U$ we have $U = U^- + G^*(H^+)$. Now we set $U^0 = U^- \cap G^*(H^+)$ and choose a closed subspace $U^+ < G^*(H^+)$ with $G^*(H^+) = U^+ \oplus U^0$. We have thus shown that the sum $U = U^+ \oplus U^-$ is direct (although, of course, not orthogonal). A standard result from functional analysis yields that the projection operators $Q^\pm : U \to U^\pm$ with respect to this sum are bounded. Furthermore, $(Q^+ - Q^-)(Q^+ - Q^-) = I$ and, therefore, the operator $Q^+ - Q^-$ is an isomorphism from $U$ onto itself. Before we can show the commutation rules (3.28) we have to show that $U^0$ is contained in the null space of $\text{Re} \, D$. To prove this we first note that from $\langle (\text{Re} \, D) G^* \psi, G^* \psi \rangle = \langle (\text{Re} \, F) \psi, \psi \rangle$ and a density argument we have
\[
\langle (\text{Re} \, D) \varphi^+, \varphi^+ \rangle \geq 0 \quad \text{for} \quad \varphi^+ \in G^*(H^+) \quad \text{and} \quad \langle (\text{Re} \, D) \varphi^-, \varphi^- \rangle \leq 0 \quad \text{for} \quad \varphi^- \in G^*(H^-)\]
Let now $\varphi^o \in U^o = U^- \cap \overline{G^+(H^+)}$. Then $\langle (\text{Re} \, D)\varphi^o, \varphi^o \rangle = 0$. For any $\varphi^+ \in \overline{G^+(H^+)}$ and any $t \in \mathbb{C}$ we conclude that $\varphi^+ + t\varphi^o \in \overline{G^+(H^+)}$ and thus

$$0 \leq \langle (\text{Re} \, D)(\varphi^+ + t\varphi^o), \varphi^+ + t\varphi^o \rangle = \langle (\text{Re} \, D)\varphi^+, \varphi^+ \rangle + 2\text{Re} \left[t \langle (\text{Re} \, D)\varphi^o, \varphi^+ \rangle\right].$$

Since this holds for all $t \in \mathbb{C}$ we conclude that $\langle (\text{Re} \, D)\varphi^o, \varphi^+ \rangle = 0$. The same argument shows that also $\langle (\text{Re} \, D)\varphi^o, \varphi^- \rangle = 0$ for all $\varphi^- \in U^-$ and thus $\langle (\text{Re} \, D)\varphi^o, \varphi \rangle = 0$ for all $\varphi \in U$. This proves $(\text{Re} \, D)\varphi^o = 0$.

Now the commutation rules (3.28) follow easily. Indeed, for $\psi \in H$ we have

$$G^* P^+ \psi = Q^+ G^* \psi = Q^+ \underbrace{G^* P^- \psi}_{\in U^-} + Q^- \underbrace{G^* P^+ \psi}_{\in \overline{G^+(H^+)}} = Q^+ G^* \psi + Q^- (\varphi^+ + \varphi^o)$$

where $G^* P^+ \psi = \varphi^+ + \varphi^o$ with $\varphi^+ \in U^+$ and $\varphi^o \in U^o \subset U^-$. Therefore, $G^* P^+ \psi = Q^+ G^* \psi + \varphi^o$ which proves (3.28) for “+” since $(\text{Re} \, D)\varphi^o = 0$. An analogous argument proves it for “−”.

Therefore, we have shown the factorization of $F_\#$ in the form (3.29), i.e.

$$F_\# = GD_\# G^* \quad \text{with} \quad D_\# := (\text{Re} \, D) \left(Q^+ - Q^-\right) + \text{Im} \, D. \quad (3.30)$$

In the second part we show now that $D_\#$ is coercive. First, we note that $F_\#$ is the sum of the non-negative operators $(\text{Re} \, D) \left(Q^+ - Q^-\right)$ and $\text{Im} \, D$. Indeed, $(\text{Re} \, D) \left(Q^+ - Q^-\right)$ is non-negative since

$$\langle (\text{Re} \, D) \left(Q^+ - Q^-\right) G^* \psi, G^* \psi \rangle = \langle G(\text{Re} \, D) G^* (P^+ - P^-) \psi, \psi \rangle = \langle |F| \psi, \psi \rangle \geq 0,$$

and the range of $G^*$ is dense. By assumption (d) of the theorem at least one of the operators is positive. Indeed, if $\text{Re} \, D$ is one-to-one then also $(\text{Re} \, D) \left(Q^+ - Q^-\right)$ which is the sum of the isomorphism $Q^+ - Q^-$ and the compact operator $K(Q^+ - Q^-)$. A standard result from functional analysis yields the existence of a constant $\bar{c} > 0$ with

$$\langle D_\# \varphi, \varphi \rangle \geq \bar{c} \| \varphi \|^2 \quad \text{for all} \quad \varphi \in U.$$

Finally, we write the factorization (3.18) in the form

$$F_\# = GD_\# G^* = \left(GD_\#^{1/2}\right) \left(GD_\#^{1/2}\right)^*$$

and proceed exactly as in the proof of Theorem 2.5. \qed

**References**


