AN INTEGRAL EQUATION APPROACH AND THE INTERIOR TRANSMISSION PROBLEM FOR MAXWELL’S EQUATIONS

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Abstract. In the first part of this paper we recall the direct scattering problem for time harmonic electromagnetic fields where arbitrary incident fields are scattered by a medium described by a space dependent permittivity, permeability, and conductivity. We present an integral equation approach and recall its basic features. In the second part we investigate the corresponding interior transmission eigenvalue problem and prove that the spectrum is discrete. Finally, we study the inhomogeneous interior transmission problem and show that it is uniquely solvable provided \( k^2 \) is not an interior eigenvalue.

1. Introduction

The investigation of eigenvalue problems play an important role in mathematical physics and applied mathematics. Often they describe resonance phenomena. As the most famous classical eigenvalue problem for partial differential equations we mention the spectrum of the (negative) Laplacian \(-\Delta\) in some open and bounded region \( D \subset \mathbb{R}^3 \) with respect to Dirichlet boundary conditions. It is well known that the spectrum consists only of a countable set of real and positive eigenvalues \( k^2 \) which converge to infinity. The corresponding eigenfunctions satisfy

\[
\Delta u + k^2 u = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D.
\]

The functional analytic approach of this kind of eigenvalue problem – in particular the question of existence of eigenvalues – uses heavily the fact that the underlying operator is self adjoint.

At first glance it is surprising that eigenvalue problems play also an important role in the investigation of scattering problems – which are special cases of exterior boundary value problems. Considering again the simplest case of Dirichlet boundary conditions the scattering of (acoustic) time harmonic waves of fixed frequency \( \omega > 0 \) by an acoustically perfectly soft obstacle \( D \) leads to an exterior Dirichlet boundary value problem which is always uniquely solvable – independent of whether or not \( k^2 = \omega^2 c_0^2 \) (where \( c_0 > 0 \) denotes the speed of sound) is an eigenvalue of \(-\Delta\).

However, in the investigation of the far field patterns \( u^\infty = u^\infty(\hat{x}, \theta) \) and the corresponding far field operator \( F \) the eigenvalue problem (1) plays an important role. We recall that the far field pattern \( u^\infty \) is the first coefficient in the asymptotic expansion of the scattered field in terms of \( r = |x| \). It depends on the angle \( \hat{x} \in S^2 \) of observation and the angle \( \theta \in S^2 \) of the incident plane wave. Here and in the

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following $S^2$ denotes the unit sphere in $\mathbb{R}^3$. For example, the far field operator $F$ from $L^2(S^2)$ into itself, defined by

\begin{equation}
(Fg)(\hat{x}) = \int_{S^2} u^\infty(\hat{x}, \theta) g(\theta) \, ds(\theta), \quad \hat{x} \in S^2,
\end{equation}

is, in general, only injective with dense range if $k^2$ is not an eigenvalue in the sense of (1).

One of the main tasks in inverse scattering theory is to determine the shape $D$ by the knowledge of $u^\infty(\hat{x}, \theta)$ for all $\hat{x}, \theta \in S^2$, i.e., equivalently, by the knowledge of the far field operator $F$. In contrast to iterative methods which try to match the given far field pattern by a sequence of far field patterns corresponding to scattering problems for iteratively improved domains $D_n$ there exist a number of methods which avoid the computation of direct scattering problems. As examples we mention Ikehata’s probe method [15, 16], Potthast’s singular sources method (see [26]), the reciprocity gap principle [1, 3], the linear sampling method (first suggested in [6], see [2]), or the factorization method [18, 21]. In many cases, the mathematical justification of these methods require the investigation of the corresponding interior problem – or the assumption that $k^2$ is not an eigenvalue. From the practical point of view such an assumption is only acceptable if there are not “too many” eigenvalues, i.e., in mathematical terms, that the set of eigenvalues is at most countable.

The scattering of acoustic waves by impenetrable obstacles $D$ with respect to other kinds of boundary conditions require the investigation of the eigenvalue problem for $-\Delta$ in $D$ with respect to the corresponding boundary conditions. An interesting observation, first discussed by David Colton and the author in [4, 17], was that the scattering of waves by penetrable obstacles $D$ lead to the so called interior transmission eigenvalue problems. If we consider the case of the scattering of acoustic waves by an inhomogeneous medium $D$ with speed of sound $c_i = c_i(x) \neq c_0$ in $D$ then one has to study the following eigenvalue problem which corresponds to (1):

\begin{equation}
\Delta v + k^2 n(x)v = 0 \text{ in } D, \quad \Delta w + k^2 w = 0 \text{ in } D,
\end{equation}

\begin{equation}
\frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \text{ on } \partial D
\end{equation}

where $n(x) = c_i^2/c_0^2(x)$ and $\nu = \nu(x)$, denotes the exterior unit normal at $x \in \partial D$. For this problem it is still unknown whether or not eigenvalues exist. Only for the spherically stratified case, i.e., the case where $D$ is a ball and $c_i$ depends only on $r = |x|$ this question has been answered positively in [8] (see [7]). For the general case, in [5] conditions on $c_i$ and $\partial D$ are given under which the set of eigenvalues is discrete (see also [7]).

It is obvious that the treatment of electromagnetic inverse scattering problems by the beforementioned methods require the investigation of the corresponding interior transmission problem for Maxwell’s equations. For example, in [19] the factorization method for non-magnetic materials has been analyzed and it had to be assumed that $k^2$ was not an interior transmission eigenvalue. In [13, 12] the corresponding interior transmission boundary value problem played an important role.

In this paper we will study the general case where the electromagnetic parameters (permittivity, permeability, conductivity) are essentially bounded scalar functions.
As seen from the analysis, the treatment of the anisotropic case should cause no additional difficulties.)

We will follow the general idea presented in [5, 7] for the scalar eigenvalue problem (3), (4). This approach transforms the eigenvalue problem into a volume integral equation for the difference \( u = v - w \) which still contains the second field \( w \). By Green’s theorem it is easily seen that \( u \) is orthogonal to the space \( V \) of all solutions \( \psi \) of the Helmholtz equation \( \Delta \psi + k^2 \psi = 0 \) in \( D \). Therefore, projecting the integral equation onto the orthogonal complement \( V^\perp \) of \( V \) leads to a homogeneous equation of the form \( u - P_k K_k u = 0 \) in \( D \) where \( K_k \) is compact and the operators \( K_k \) and \( P_k \) depend non-linearly on \( k \). The main work has to be put into the proof that they depend also analytically on \( k \) which allows the application of the analytic Fredholm theory.

In order to treat the electromagnetic case by the same method it is necessary to develop an appropriate integral equation method for scattering problems. This will be done in Section 2. Although this approach is certainly known to the experts the author was not able to find it in the (mathematical) literature. The integral equations method presented in [7] is restricted to non-magnetic media and smooth permittivity and conductivity in order to derive an integral equation with weakly singular kernels. The integral equation presented in this paper – which is actually an integro-differential equation – holds for general \( L^\infty \) parameters \( \varepsilon, \mu, \) and \( \sigma \) and is particularly simple for non-magnetic materials. It contains, however, strongly singular kernels. We want to point out that in our opinion the variational form and the integral equation method complement each other in a very fruitful way which will be seen clearly from our approach. One advantage of the integral equation method is the simple derivation of the electromagnetic far field patterns. To motivate again the study of the interior transmission eigenvalue problem at the end of Section 2 we introduce the far field operator and show that it is one-to-one if \( k^2 \) is not an eigenvalue.

Section 3 is devoted to the study of the interior transmission problem. As the final result of Subsection 3.1 we will prove that the spectrum is discrete. The proof is considerably more difficult than in the scalar case. Finally, we will treat the inhomogeneous equation in Subsection 3.2.

2. An Integral Equation Method for the Scattering Problem

In this section we study the direct scattering problem. Let \( k = \omega \sqrt{\varepsilon\mu_0} > 0 \) be the wave number with frequency \( \omega \), electric permittivity \( \varepsilon_0 \), and magnetic permeability \( \mu_0 \) in vacuum. An incident electromagnetic field consists of a pair \( H^i \) and \( E^i \) which satisfy the time harmonic Maxwell system in vacuum, i.e.

\[
\text{curl } E^i - i\omega \mu_0 H^i = 0 \quad \text{in } \mathbb{R}^3,
\]

\[
\text{curl } H^i + i\omega \varepsilon_0 E^i = 0 \quad \text{in } \mathbb{R}^3.
\]

This incident wave is scattered by a medium with space dependent electric permittivity \( \varepsilon = \varepsilon(x) \), magnetic permeability \( \mu = \mu(x) \), and conductivity \( \sigma = \sigma(x) \). We assume that \( \varepsilon \equiv \varepsilon_0 \) and \( \mu \equiv \mu_0 \) and \( \sigma \equiv 0 \) outside of some bounded domain. The total fields are superpositions of the incident and scattered fields, i.e. \( E = E^i + E^s \) and \( H = H^i + H^s \) and satisfy the Maxwell system

\[
\text{curl } E - i\omega \mu H = 0 \quad \text{in } \mathbb{R}^3,
\]
curl \( H + i\omega \varepsilon E = \sigma E \) in \( \mathbb{R}^3 \).

Furthermore, the tangential components of \( E \) and \( H \) are continuous on interfaces where \( \sigma, \varepsilon, \) or \( \mu \) are discontinuous. Finally, the scattered fields have to satisfy the Silver-Müller radiation condition

\[
\sqrt{\frac{\mu_0}{\varepsilon_0}} H^s(x) \times x - |x| E^s(x) = O \left( \frac{1}{|x|} \right) \quad \text{as } |x| \to \infty
\]

uniformly with respect to \( \hat{x} = x/|x| \).

In this paper we will always work with the magnetic field \( H \) only. This is motivated by the fact that for the important case of non-magnetic media (i.e. \( \mu \equiv \mu_0 \)) the magnetic field is divergence free as seen from (7) and the fact that \( \text{div} \, \text{curl} = 0 \). However, as seen from (7), (8) the analysis works equally well for the electric field if we interchange the roles of \( -\mu \) and \( \varepsilon + i\sigma/\omega \).

Therefore, eliminating the electric field \( E \) from the system (7), (8) leads to

\[
curl \left[ \frac{1}{\sigma - i\omega \varepsilon} \text{curl} H \right] - i\omega \mu H = 0,
\]

i.e.,

\[
\text{curl} \left[ \frac{1}{\varepsilon_r} \text{curl} H \right] - k^2 \mu_r H = 0 \quad \text{in} \ \mathbb{R}^3
\]

where \( \varepsilon_r \) denotes the (complex valued) relative permittivity and \( \mu_r \) the relative magnetic permeability given by

\[
\varepsilon_r(x) = \frac{\varepsilon(x)}{\varepsilon_0} + i \frac{\sigma(x)}{\omega \varepsilon_0}, \quad \mu_r(x) = \frac{\mu(x)}{\mu_0},
\]

respectively. We note that \( \varepsilon_r \equiv 1 \) and \( \mu_r \equiv 1 \) outside of some bounded domain. The incident field \( H^i \) satisfies

\[
\text{curl}^2 H^i - k^2 H^i = 0 \quad \text{in} \ \mathbb{R}^3.
\]

Subtracting equations (10) and (12) yields for the scattered field

\[
\text{curl} \left[ \frac{1}{\varepsilon_r} \text{curl} H^s \right] - k^2 \mu_r H^s = k^2 p H^i + \text{curl} \left[ q \text{curl} H^i \right] \quad \text{in} \ \mathbb{R}^3,
\]

where the contrasts \( p \) and \( q \) are defined by \( p = \mu_r - 1 \) and \( q = 1 - 1/\varepsilon_r \), respectively. The Silver-Müller radiation condition turns into

\[
\text{curl} H^s(x) \times \hat{x} - ik H^s(x) = O \left( \frac{1}{|x|^2} \right), \quad |x| \to \infty.
\]

The continuity of the tangential components of \( E \) and \( H \) translates into analogous requirements for \( H^s \) and \( \text{curl} H^s \).

We will allow more general source terms on the right-hand side of (13). In particular, we will consider the following problem: Given a bounded domain \( D \subset \mathbb{R}^3 \) and \( g, h \in L^2(D, \mathbb{C}^3) \) and \( \mu_r, \varepsilon_r \in L^\infty(D) \) and \( k \in \mathbb{C} \setminus \{0\} \) with \( \text{Re} k \geq 0, \text{Im} k \geq 0 \) determine \( v \) such that

\[
\text{curl} \left[ \frac{1}{\varepsilon_r} \text{curl} v \right] - k^2 \mu_r v = k^2 h + \text{curl} g \quad \text{in} \ \mathbb{R}^3
\]

\[\text{By } L^2(D, \mathbb{C}^3), \text{ etc, we denote the vector fields from } D \text{ into } \mathbb{C}^3 \text{ such that each component is in } L^2(D).\]
In the following, we assume that
\( \partial D \) in the space as of (10) and (13) have to be understood in the variational sense, i.e., are sought

\[
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\]

\( \equiv \mu \) can be proven by the Riesz-Fredholm theory. However, this approach assumes that
an equivalent integral equation for the electric field is derived from which existence
involving the exterior Calderon operator. A similar – but different – approach has
problem (5) – (9) or (15) – (17). In [22] it was suggested to transform (18) into
Classical interior regularity results (cf. [14]) yield that \( \psi \) for all
Here we denote by
\( H \)
where \( \varepsilon \)
uniformly with respect to \( \hat{x} \in S^2 \). Here – and always in the following – we extend \( \varepsilon_r \) and \( \mu_r \) by one and \( f, g \) by zero outside of \( D \). The solutions \( v \) of (15), (16) as well as of (10) and (13) have to be understood in the variational sense, i.e., are sought in the space

\[
H_{loc}(\text{curl}, \mathbb{R}^3) = \{ v : \mathbb{R}^3 \to \mathbb{C}^3 : v|_B \in H(\text{curl}, B) \text{ for all balls } B \subseteq \mathbb{R}^3 \}
\]

where \( H(\text{curl}, B) \) is the completion of \( C^\infty(B) \) with respect to the norm induced by the inner product

\[
\langle v, w \rangle_{H(\text{curl}, B)} := \langle v, w \rangle_{L^2(B)} + \langle \text{curl } v, \text{curl } w \rangle_{L^2(B)}.
\]

Here we denote by \( \langle \cdot, \cdot \rangle_{L^2(B)} \) the inner product in \( L^2(B, \mathbb{C}^3) \). Then \( v \in H_{loc}(\text{curl}, \mathbb{R}^3) \) is said to be the variational solution of (15), (16) if \( v \) satisfies

\[
\int_{\mathbb{R}^3} \left[ \frac{1}{\varepsilon_r} \text{curl } v \cdot \text{curl } \psi - k^2 \mu_r v \cdot \psi \right] dx = \int_D \left[ k^2 h \cdot \psi + g \cdot \text{curl } \psi \right] dx
\]

for all \( \psi \in H(\text{curl}, \mathbb{R}^3) \) with compact support. Outside of \( D \) the solution satisfies
\( \text{curl}^2 v - k^2 v = 0 \). Taking the divergence of this equation and using the identities
\( \text{div} \text{curl} = 0 \) and \( \text{curl}^2 = -\Delta + \nabla \text{div} \) this system is equivalent to the pair of equations

\[
\Delta v + k^2 v = 0 \quad \text{and} \quad \text{div } v = 0.
\]

Classical interior regularity results (cf. [14]) yield that \( v \) is analytic outside of \( D \). In particular, the radiation condition (17) is well defined.

There are several approaches for proving existence of a solution of the scattering problem (5) – (9) or (15) – (17). In [22] it was suggested to transform (18) into a variational equation on a bounded domain with non-local boundary conditions involving the exterior Calderon operator. A similar – but different – approach has been studied in [11] for the case of a layered background medium, cf. also [24]. In [7] an equivalent integral equation for the electric field is derived from which existence can be proven by the Riesz-Fredholm theory. However, this approach assumes that \( \mu \equiv \mu_0 \) and \( \sigma \) and \( \varepsilon_r \) are smooth functions in \( \mathbb{R}^3 \), and it is not clear how this method can be generalized to non-smooth data. In [20] we derived a new integral equation (for non-magnetic media but even for the case where the background medium is layered) which we will extend in the following to magnetic media.

**Assumptions 2.1**. In the following we assume that \( D \subset \mathbb{R}^3 \) is open and bounded with \( \partial D \in C^{2,\alpha} \). Furthermore, let \( k \in \mathbb{C} \setminus \{0\} \) with \( \text{Re } k \geq 0 \) and \( \text{Im } k \geq 0 \) be the wave number and \( \varepsilon_r, \mu_r \in L^\infty(\mathbb{R}^3) \) such that \( \mu_r \) is real valued and \( \text{Im } \varepsilon_r \geq 0 \) and \( \mu_r \equiv \varepsilon_r \equiv 1 \) on \( \mathbb{R}^3 \setminus D \). Furthermore, we assume that there exists \( c_0 > 0 \) with \( \text{Re } \varepsilon_r \geq c_0 \) on \( D \) and \( \mu_r \geq c_0 \) on \( D \). Then, in particular, \( 1/\varepsilon_r \in L^\infty(\mathbb{R}^3) \).

The following lemma is the key ingredient of the integral equation method.

\[
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\]
Lemma 2.2. Let the fundamental solution $\Phi_k$ of the scalar Helmholtz equation in $\mathbb{R}^3$ be defined by

$$\Phi_k(x, y) = \frac{\exp(ik|x-y|)}{4\pi |x-y|}, \quad x \neq y.$$ 

(a) For $\varphi \in L^2(D)$ the (scalar) Riesz potential

$$w(x) = \iint_D \varphi(y) \Phi_k(x, y) \, dy, \quad x \in \mathbb{R}^3,$$

defines a function in $H^2_{loc}(\mathbb{R}^3)$ which satisfies $\Delta w + k^2 w = -\varphi$ in $\mathbb{R}^3$ where we extended $\varphi$ by zero into $\mathbb{R}^3$. Furthermore, $w$ satisfies the Sommerfeld radiation condition (see [7]), and the restriction $w|_D$ of $w$ to $D$ defines a bounded operator from $L^2(D)$ into $H^2(D)$.

(b) For $g \in L^2(D, \mathbb{C}^3)$ the vector field

$$u(x) = \text{curl} \iint_D g(y) \Phi_k(x, y) \, dy, \quad x \in \mathbb{R}^3,$$

defines a function in $H^2_{loc}(\text{curl}, \mathbb{R}^3)$ which satisfies $\text{curl}^2 u - k^2 u = \text{curl} g$ in the variational sense, i.e.

$$\iint_{\mathbb{R}^3} \left[ \text{curl} u \cdot \text{curl} \psi - k^2 u \cdot \psi \right] dx = \iint_D g \cdot \text{curl} \psi \, dx$$

for all $\psi \in H(\text{curl}, \mathbb{R}^3)$ with compact support. Furthermore, $u$ satisfies the Silver-Müller radiation condition (17), and the restriction $u|_D$ of $u$ to $D$ defines a bounded operator from $L^2(D, \mathbb{C}^3)$ into $H(\text{curl}, D)$.

(c) For $g \in L^2(D, \mathbb{C}^3)$ the vector field

$$u(x) = (k^2 + \nabla \text{div}) \iint_D g(y) \Phi_k(x, y) \, dy, \quad x \in \mathbb{R}^3,$$

defines a function in $H^2_{loc}(\text{curl}, \mathbb{R}^3)$ which satisfies $\text{curl}^2 u - k^2 u = k^2 g$ in the variational sense. Furthermore, $u$ satisfies the Silver-Müller radiation condition (17), and the restriction $u|_D$ of $u$ to $D$ defines a bounded operator from $L^2(D, \mathbb{C}^3)$ into $H(\text{curl}, D)$.

Proof. Part (a) is well known, see [14]. For part (b) we refer to [20].

(c) By (a) we note that $g \mapsto u|_D$ is bounded from $L^2(D, \mathbb{C}^3)$ into itself. Furthermore, since $\text{curl} \nabla = 0$,

$$\text{curl}^2 u(x) = k^2 \text{curl} \iint_D g(y) \Phi_k(x, y) \, dy$$

$$= k^2 \left[ \nabla \text{div} - (\Delta + k^2) + k^2 \right] \iint_D g(y) \Phi_k(x, y) \, dy$$

$$= k^2 \left[ \nabla \text{div} + k^2 \right] \iint_D g(y) \Phi_k(x, y) \, dy + k^2 g(x)$$

$$= k^2 u(x) + k^2 g(x)$$

where we have used part (a) again. Finally, from $\text{curl} u = k^2 \text{curl} \iint_D g(y) \Phi_k(\cdot, y) \, dy$ and part (b) we receive boundedness of $g \mapsto u|_D$ from $L^2(D, \mathbb{C}^3)$ into $H(\text{curl}, D)$. □
Let (a) We write (15) in the form equation (23).

Proof. Substituting the forms of \( v \) and \( \psi \) into (15), (16). Then

\[
\int D \left[ \text{curl} \cdot \text{curl} - k^2 v \cdot \psi \right] dx = \int D \left[ k^2 (h + p v) \cdot \psi + (g + q \text{curl} v) \cdot \text{curl} \psi \right] dx
\]

for all \( \psi \in H(\text{curl}, \mathbb{R}^3) \) with compact support. Again, we recall that \( p = \mu_r - 1 \) and \( q = 1 - 1/\varepsilon_r \).

We will now prove that this problem is equivalent to the following integro-differential equation:

\[
(23) \quad v(x) = (k^2 + \nabla \text{div}) \int D \left[ p(y) v(y) + h(y) \right] \Phi_k(x, y) dy + \\
+ \text{curl} \int D \left[ g(y) \text{curl} v(y) + g(y) \right] \Phi_k(x, y) dy, \quad x \in D.
\]

Theorem 2.3. Let \( k \in \mathbb{C} \setminus \{0\} \) with Re \( k \geq 0 \) and Im \( k \geq 0 \).

(a) Let \( v \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3) \) be a radiating solution of (15), (16) in the sense of (18). Then \( v|_D \in H(\text{curl}, D) \) solves (23).

(b) Let \( v \in H(\text{curl}, D) \) solve (23). Then \( v \) can be extended by the right side of (23) to a radiating solution of (15), (16).

Proof. (a) We write (15) in the form

\[
\text{curl}^2 v - k^2 v = k^2 [p v + h] + \text{curl} [q \text{curl} v + g]
\]

and define \( v_1 \) and \( v_2 \) by

\[
v_1(x) = (k^2 + \nabla \text{div}) \int D \left[ p(y) v(y) + h(y) \right] \Phi_k(x, y) dy, \quad x \in \mathbb{R}^3,
\]

\[
v_2(x) = \text{curl} \int D \left[ g(y) \text{curl} v(y) + g(y) \right] \Phi_k(x, y) dy, \quad x \in \mathbb{R}^3.
\]

By Lemma 2.2 we conclude that \( v_1, v_2 \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3) \) and

\[
\text{curl}^2 v_1 - k^2 v_1 = k^2 [p v + h], \quad \text{curl}^2 v_2 - k^2 v_2 = \text{curl} [q \text{curl} v + g]
\]

in the variational sense. Therefore, by (15),

\[
\text{curl}^2 (v_1 + v_2) - k^2 (v_1 + v_2) = k^2 [p v + h] + \text{curl} [q \text{curl} v + g] = \text{curl}^2 v - k^2 v \quad \text{in } \mathbb{R}^3.
\]

Since both, \( v_1 + v_2 \) and \( v \), satisfy the radiation condition we conclude that \( v = v_1 + v_2 \) in \( \mathbb{R}^3 \). Substituting the forms of \( v_1 \) and \( v_2 \) into \( v = v_1 + v_2 \) yields that \( v \) satisfies equation (23).

(b) Let \( v \in H(\text{curl}, D) \) be a solution of (23). Extend \( v \) by the right hand side of (23) to all of \( \mathbb{R}^3 \). By using Lemma 2.2 again we conclude that

\[
\text{curl}^2 v - k^2 v = k^2 [p v + h] + \text{curl} [q \text{curl} v + g],
\]

i.e. \( v \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3) \) satisfies (15), (16). \( \square \)
From the extension of the integro-differential equation (23) to the exterior of \( D \) we see immediately that every radiating solution \( v \) of (15), (16) has an asymptotic behaviour of the form

\[
v(x) = \frac{\exp(ik|x|)}{4\pi|x|} v^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,
\]

uniformly with respect to \( \hat{x} = x/|x| \). The vector field \( v^\infty \) is called the far field pattern of \( v \). It is an analytic function on the unit sphere \( S^2 = \{ x \in \mathbb{R}^3 : |x| = 1 \} \) with respect to \( \hat{x} \) and is a tangential field, i.e. satisfies \( v^\infty(\hat{x}) \cdot \hat{x} = 0 \) for all \( \hat{x} \in S^2 \).

To study solvability of (23) we define the operators \( A_k \) and \( B_k \) of \( H(\text{curl}, D) \) into itself and the function \( f \in H(\text{curl}, D) \) by

\[
(A_k v)(x) = (k^2 + \nabla \text{div}) \int_D p(y)v(y)\Phi_k(x, y) \, dy, \quad x \in D,
\]

\[
(B_k v)(x) = \text{curl} \int_D q(y) \text{curl} v(y)\Phi_k(x, y) \, dy, \quad x \in D,
\]

\[
f(x) = (k^2 + \nabla \text{div}) \int_D h(y)\Phi_k(x, y) \, dy + \text{curl} \int_D g(y)\Phi_k(x, y) \, dy, \quad x \in D.
\]

Then (23) can be written as \( v - A_k v - B_k v = f \). We note that \( B_k \) is bounded from \( H(\text{curl}, D) \) into itself and \( A_k \) is bounded from \( L^2(D, \mathbb{C}^3) \) into \( H(\text{curl}, D) \) (Lemma 2.2, parts (b), (c)). Also, we note that \( f \in H(\text{curl}, D) \).

The operator on the left hand side of \( v - A_k v - B_k v = f \) is Fredholm of index zero by the following result.

**Lemma 2.4.** (a) For \( k = i \) and any \( f \in H(\text{curl}, D) \) the equation \( v - A_i v - B_i v = f \) is uniquely solvable in \( H(\text{curl}, D) \) and the solution \( v \) depends continuously on \( f \). In other words: The operator \( I - A_i - B_i \) is boundedly invertible in \( H(\text{curl}, D) \).

(b) The operators \( A_k - A_i \) and \( B_k - B_i \) are compact in \( H(\text{curl}, D) \).

**Proof.** We consider the equation \( w - A_i w - B_i w = A_i f + B_i f \) which is equivalent to \( v - A_i v - B_i v = f \) by setting \( w = v - f \). By inspection of the right hand side of the equation \( w - A_i w - B_i w = A_i f + B_i f \) we note that it is of the form (23) for \( k = i \) and \( h = pf \) and \( g = q \text{curl} f \). Application of Theorem 2.3 for \( k = i \) and this choice of \( h \) and \( g \) yields that \( w - A_i w - B_i w = A_i f + B_i f \) is equivalent to the radiation problem

\[
\text{curl} \left( \frac{1}{\varepsilon_r} \text{curl} w \right) + \mu_r w = -pf + \text{curl}[q \text{curl} f] \quad \text{in } \mathbb{R}^3.
\]

The variational form of this equation is (replacing \( \psi \) by its complex conjugate \( \overline{\psi} \)),

\[
\iint_{\mathbb{R}^3} \left[ \frac{1}{\varepsilon_r} \text{curl} w \cdot \text{curl} \overline{\psi} + \mu_r w \cdot \overline{\psi} \right] dx = \iint_D \left[ -pf \cdot \overline{\psi} + q \text{curl} f \cdot \text{curl} \overline{\psi} \right] dx
\]
Let Assumption 2.1 hold. Part (a) is shown by combining Theorem 2.3 and Lemma 2.4. Let, in addition,\\n\[ v \text{ ds} \]
and a function\\n\\n\[ \varphi \]
\[ = \]
\[ (23) \] yield
\[ \psi \]
and the definition of \( \Phi \). Theorem 2.5. The left hand side defines a coercive sesqui-linear form on \( H(\text{curl}, \mathbb{R}^3) \), and the right hand side a bounded conjugate-linear functional on \( H(\text{curl}, \mathbb{R}^3) \). The theorem of Lax-Milgram yields the existence of a unique solution \( w \) of the variational equation.

(b) The first part of the operator \( A_k \) is compact by the weak singularity of \( \Phi_k \) and \( \nabla \Phi_k \). The kernel in the second part of \( A_k - A_i \) is the Hessian matrix
\[ \nabla^2 [\Phi_k(x, y) - \Phi_i(x, y)] \in \mathbb{C}^{3 \times 3} \] which has a singularity as \( 1/|x - y| \). Furthermore, the curl of the second part vanishes. This shows compactness of \( A_k - A_i \). With the same arguments one observes that also \( B_k - B_i \) is compact.

Proof. Part (a) is shown by combining Theorem 2.3 and Lemma 2.4.

For part (b) we assume that \( v \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3) \) is a radiating solution with vanishing \( g \) and \( h \). We choose an open ball \( B_R \) with radius \( R \) such that \( \overline{D} \subset B_R \) and a function \( \phi \in C^\infty(\mathbb{R}^3) \) with compact support and \( \phi \equiv 1 \) on \( B_R \). Setting \( \psi = \varphi \phi \) in (18) yields
\[ \frac{1}{|x|} \left[ |\text{curl } v|^2 - k^2 \mu_r |v|^2 \right] \, dx + \int_{|x| > R} \left[ \text{curl } v \cdot \text{curl } \psi - k^2 v \cdot \psi \right] \, dx = 0. \]

Applying Green’s theorem to the second integral yields
\[ \int_{|x| < R} \frac{1}{|x|} |\text{curl } v|^2 - k^2 \mu_r |v|^2 \, dx + \int_{|x| = R} (\nu \times \text{curl } v) \cdot \varphi \, ds = 0. \]

The radiating condition (17) and the behaviour \( v(x) = \mathcal{O}(1/|x|) \) (also seen from (23)) yield
\[ \int_{|x| < R} \frac{1}{|x|} |\text{curl } v|^2 - k^2 \mu_r |v|^2 \, dx = ik \int_{|x| = R} |v|^2 \, ds + \mathcal{O}(1/R) \]

(30)
\[ = ik \int_{S^2} |v|^2 \, ds + \mathcal{O}(1/R) \]

\[ \text{(b) Actually, we have shown only one part of it.} \]
as \( R \) tends to infinity. Here we have also used the definition (24) of the far field pattern \( v^\infty \). Now we take the imaginary part and observe that \( \text{Im}(1/\varepsilon_r) = -\text{Im} \varepsilon_r/|\varepsilon_r|^2 \leq 0 \). Therefore, letting \( R \) tend to infinity we conclude that \( v^\infty \) vanishes and thus also \( v \) in the exterior of \( D \) by Rellich’s Lemma and unique continuation (see [7], note that here we need the connectedness of the exterior of \( D \)).

In the case where \( \text{Im} \varepsilon > 0 \) on \( D \) we conclude from the imaginary part of (30) that \( \text{curl} \ v \) vanishes in \( D \). The variational equation (18) (which reduces now to one for \( v \) in \( D \)) yields that also \( v \) vanishes in \( D \).

In the case where \( \mu_r \) and \( \varepsilon_r \) are smooth we first extend \( \mu_r \) and \( \varepsilon_r \) to \( \mu_r \in C^{2,\alpha}(\mathbb{R}^3) \) and \( \varepsilon_r \in C^{1,\alpha}(\mathbb{R}^3) \), respectively. Now we argue as in [7] and compute \( \text{curl}^2 v \) from (15) as

\[
\text{curl}^2 v = -\varepsilon_r \nabla \frac{1}{\varepsilon_r} \times \text{curl} v + k^2 \mu_r \varepsilon_r v.
\]

From \( \text{div} (\mu_r v) = 0 \) one gets \( \text{div} v = - (\nabla \mu_r)/\mu_r \cdot v \) and thus

\[
\Delta v = -\nabla \left( \frac{\nabla \mu_r}{\mu_r} \cdot v \right) + \varepsilon_r \nabla \frac{1}{\varepsilon_r} \times \text{curl} v - k^2 \mu_r \varepsilon_r v.
\]

The unique continuation principle (Lemma 8.5 in [7]) can be applied and yields that \( v \) vanishes in \( D \).

In particular, under the assumptions (b) of the previous theorem the scattering problem (5) – (9) is uniquely solvable for any incident fields \( H^i \) and \( E^i \). In order to give one of several motivations for the study of the interior eigenvalue problem we consider the special case where the incident waves \( H^i \) and \( E^i \) are given by plane waves of the form

\[
H^i(x; \theta, g) = g e^{ik_\theta \cdot x} \quad \text{and} \quad E^i(x; \theta, g) = -\frac{1}{i\omega \varepsilon_0} \text{curl} H^i(x; \theta, g) = -\sqrt{\mu_0/\varepsilon_0} (\theta \times g) e^{ik_\theta \cdot x}.
\]

Here, \( \theta \in S^2 \) denotes the direction of incidence and \( g \in \mathbb{C}^3 \) the polarization vector. We have to assume that \( g \cdot \theta = \sum_{j=1}^3 g_j \theta_j = 0 \) in order to ensure that \( H^i \) is divergence free. Then the far field patterns \( H^\infty \) and \( E^\infty \) of \( H^i \) and \( E^i \), respectively, depend on \( \theta \) and \( g \) as well. We will work only with \( H^\infty \) and indicate this dependence by writing \( H^\infty = H^\infty(\hat{x}; \theta, g) \). Note again, that \( H^\infty \) is a tangential vector field, i.e. it satisfies \( H^\infty(\hat{x}; \theta, g) \cdot \hat{x} = 0 \) for all \( \hat{x} \in S^2 \) and all \( \theta \in S^2 \) and \( g \in \mathbb{C}^3 \) with \( g \cdot \theta = 0 \).

Introducing the subspace \( L^2(S^2) \) of \( L^2(S^2, \mathbb{C}^3) \) by

\[
L^2(S^2) := \{ v \in L^2(S^2, \mathbb{C}^3) : v(\hat{x}) \cdot \hat{x} = 0, \; \hat{x} \in S^2 \}
\]

we note that the far field operator \( F : L^2(S^2) \rightarrow L^2(S^2) \), defined by

\[
(Fg)(\hat{x}) := \int_{S^2} H^\infty(\hat{x}; \theta, g(\theta)) \, ds(\theta), \; \hat{x} \in S^2,
\]

is well defined. One of many important properties of \( F \) is given in the following theorem:

**Theorem 2.6.** Let the assumptions (b) of Theorem 2.5 be satisfied. Then the far field operator \( F : L^2(S^2) \rightarrow L^2(S^2) \) is one-to-one provided the following interior transmission problem admits only the trivial solution \((v, w) = (0, 0)\) in \( H(\text{curl}, D) \times H(\text{curl}, D) \):

\[
\text{curl} \left[ \frac{1}{\varepsilon_r} \text{curl} v \right] - k^2 \mu_r v = 0 \; \text{in} \; D, \quad \text{curl}^2 w - k^2 w = 0 \; \text{in} \; D,
\]
Let $\partial D$ on $f \equiv \text{ satisfy equations (32), (33).}$

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Then $v^\infty = Fg = 0$ and thus by Rellich’s lemma and unique continuation (see [7]) also $v^s = 0$ outside of $D$, i.e. $v = v_g$ outside of $D$. Therefore, $v|_D$ and $w = v_g|_D$ satisfy equations (32), (33).

3. The Interior Transmission Problem

First, we extend the definition (32), (33) of an interior transmission eigenvalue problem to an interior transmission boundary value problem.

Definition 3.1. For given $f \in L^2(D, \mathbb{C}^3)$ and tangential vector fields $g \in H^{3/2}(\partial D, \mathbb{C}^3)$, and $h \in H^{1/2}(\partial D, \mathbb{C}^3)$ the interior transmission problem is to determine $(v, w) \in H(\text{curl}, D) \times H(\text{curl}, D)$ such that

(34) $\text{curl} \left[ \frac{1}{\varepsilon_r} \text{curl} v \right] - k^2 \mu_r v = 0 \text{ in } D,$ $\text{curl}^2 w - k^2 w = f \text{ in } D,$

and

(35) $\nu \times v - \nu \times w = g \text{ on } \partial D,$ $\frac{1}{\varepsilon_r} \nu \times \text{curl} v - \nu \times \text{curl} w = h \text{ on } \partial D.$

$k^2$ is called an **interior transmission eigenvalue** with corresponding eigenpair $(v, w) \in H(\text{curl}, D) \times H(\text{curl}, D)$ if $(v, w) \neq (0, 0)$ and $(v, w)$ satisfies (34), (35) for $f \equiv 0$ in $D$ and $g = h \equiv 0$ on $\partial D$.

Again, we understand (34), (35) in the variational sense, i.e. $\nu \times v - \nu \times w = g$ on $\partial D$ in the sense of traces and

(36) $\iiint_D \left[ \frac{1}{\varepsilon_r} \text{curl} v \cdot \text{curl} \psi - k^2 \mu_r v \cdot \psi \right] dx = - \iiint_D f \cdot \psi dx +$

$+ \iiint_D \left[ \text{curl} w \cdot \text{curl} \psi - k^2 w \cdot \psi \right] dx - \int_{\partial D} h \cdot \psi ds$

for all $\psi \in H(\text{curl}, D)$ and

(37) $\iiint_D \left[ \text{curl} w \cdot \text{curl} \psi - k^2 w \cdot \psi \right] dx = \iiint_D f \cdot \psi dx \text{ for all } \psi \in H_0(\text{curl}, D)$

where $H_0(\text{curl}, D) = \{ v \in H(\text{curl}, D) : \nu \times v = 0 \text{ on } \partial D \}.$

Using a lifting argument we can reduce this problem to one with homogeneous boundary conditions, i.e. $g = h \equiv 0$. As shown in [12], Lemma 3.1, there exists $\hat{w} \in H^2(D, \mathbb{C}^3)$ with

$\nu \times \hat{w} = g \text{ on } \partial D$ and $\nu \times \text{curl} \hat{w} = h \text{ on } \partial D.$
Then we have \( \nu \times v = \nu \times (w + \hat{w}) \) on \( \partial D \) and
\[
\iint_D \left[ \frac{1}{\varepsilon_r} \text{curl } v \cdot \text{curl } \psi - k^2 \mu_r v \cdot \psi \right] \, dx = - \iint_D f \cdot \psi \, dx + \iint_D [\text{curl } w \cdot \text{curl } \psi - k^2 w \cdot \psi] \, dx - \int_{\partial D} \psi \cdot (\nu \times \text{curl } \hat{w}) \, ds \\
= - \iint_D [f + \text{curl}^2 \hat{w} - k^2 \hat{w}] \cdot \psi \, dx + \iint_D [\text{curl } (w + \hat{w}) \cdot \text{curl } \psi - k^2 (w + \hat{w}) \cdot \psi] \, dx
\]
for all \( \psi \in H(\text{curl}, D) \) where we have used Green’s formula again. This variational form for \( v \) and \( w + \hat{w} \) now takes the form of (36) with \( f \) replaced by \( f + \text{curl}^2 \hat{w} - k^2 \hat{w} \) and \( g = h \equiv 0 \). (37) is seen analogously. Therefore, from now on we consider only the case of vanishing \( g \) and \( h \).

**Remark:** The assumptions on \( g \) and \( h \) don’t seem to be optimal since only \( \hat{w} \in H(\text{curl}^2, D) \) is required. However, in the applications for the linear sampling method \( g \) and \( h \) are the traces of Green’s tensors and their derivatives which certainly satisfy these smoothness assumptions.

Next, we show that for conducting media there exist no real eigenvalues.

**Theorem 3.2.** Let \( \text{Im } \varepsilon_r > 0 \) on \( D \). Then there exist no real eigenvalues \( k^2 \).

**Proof.** Let \( (v, w) \in H(\text{curl}, D) \times H(\text{curl}, D) \) solve (36), (37) for \( f = 0 \) and \( g = h = 0 \). Setting \( \psi = \pi \) in (36) yields
\[
\iint_D \left[ \frac{1}{\varepsilon_r} |\text{curl } v|^2 - k^2 \mu_r |v|^2 \right] \, dx = \iint_D \left[ |\text{curl } w|^2 - k^2 |w|^2 \right] \, dx \\
+ \iint_D [\text{curl } w \cdot \text{curl } (\bar{v} - \bar{w}) - k^2 w \cdot (\bar{v} - \bar{w})] \, dx \\
= \iint_D \left[ |\text{curl } w|^2 - k^2 |w|^2 \right] \, dx
\]
by (37) since \( v - w \in H_0(\text{curl}, D) \). Taking the imaginary part yields \( \text{curl } v = 0 \) in \( D \). From (36) and (37) we conclude that \( \iint_D \mu_r v \cdot \psi \, dx = 0 \) for all \( \psi \in H_0(\text{curl}, D) \) and thus also \( v = 0 \) in \( D \) by a density argument. Therefore \( w \) satisfies (37) for \( f = 0 \) and all \( \psi \in H(\text{curl}, D) \) and, in addition, \( \nu \times w = 0 \) on \( \partial D \). Extending \( w \) by zero outside of \( D \) we observe that \( w \in H(\text{curl}, \mathbb{R}^3) \) satisfies the Helmholtz equation (in the variational sense) in all of \( \mathbb{R}^3 \). A regularity result and unique continuation yields that \( w = 0 \) in \( D \).

Therefore, for the remaining part of this section we make the following general assumption on \( D, k, \mu_r \), and \( \varepsilon_r \).

**Assumptions 3.3.** Let \( D \subset \mathbb{R}^3 \) be open and bounded such that \( \partial D \in C^{2, \alpha} \). Let \( k \in \mathbb{R}_{>0} \) be the wave number and let \( \varepsilon_r, \mu_r \in L^\infty(D) \) be real valued such that \( \varepsilon_r \geq 1 + \gamma_0 \) on \( D \) and \( \mu_r \geq 1 + \gamma_0 \) on \( D \) for some \( \gamma_0 > 0 \) and \( \|\mu_r\|_{\infty} < 2 \).
We introduce the closed affine subspace $V_k$ of $H(\text{curl}, D)$ as
$$
V_k = \{ w \in H(\text{curl}, D) : \text{curl}^2 w - k^2 w = f \text{ in } D \},
$$
where the differential equation is again understood in the variational sense (37).

The corresponding subspace for $f = 0$ is denoted by $V_{k,0}$.

**Lemma 3.4.** Let $f \in L^2(D, \mathbb{C}^3)$ and set $\hat{f} = \frac{1}{\text{Re} \mu} f$ where again $p = \mu_r - 1 \geq \gamma_0 > 0$ and $q = 1 - 1/\varepsilon_r$. Then $(v, w) \in H(\text{curl}, D) \times V_k$ solves (34), (35) for $g = h = 0$ if, and only if,

$$
v = A_k v + B_k v + w - A_k \hat{f}
$$

and

$$
\iint_D [k^2 p v \cdot \overline{\psi} + q \text{curl} v \cdot \text{curl} \overline{\psi}] \, dx = \iint_D f \cdot \overline{\psi} \, dx \text{ for all } \psi \in V_{k,0}.
$$

The operators $A_k$ and $B_k$ have been introduced in (25) and (26), respectively.

**Proof.** (a) Let first $(v, w) \in H(\text{curl}, D) \times V_k$ be a solution of (34), (35) for $g = h = 0$. We set $u = v - w$ in $D$ and extend $u$ to zero outside of $D$. Then $u \in H(\text{curl}, \mathbb{R}^3)$ and $u$ is a solution of the radiating problem (15), (16) (in the sense of (18)) with $h = pw - pf$ and $g = q \text{curl} w$. By Theorem 2.3 $u$ satisfies $u - A_k u - B_k u = A_k (w - \hat{f}) + B_k w$, i.e. $(v, w)$ satisfies (38). Furthermore, from (36) we conclude that $u = v - w$ satisfies

$$
\iint_D [\text{curl} u \cdot \text{curl} \overline{\psi} - k^2 u \cdot \overline{\psi}] \, dx = - \iint_D f \cdot \overline{\psi} \, dx + \iint_D [k^2 p v \cdot \overline{\psi} + q \text{curl} v \cdot \text{curl} \overline{\psi}] \, dx
$$

for all $\psi \in H(\text{curl}, D)$. The left hand side vanishes for $\psi \in V_{k,0}$ by (37) since $u \in H_0(\text{curl}, D)$. This proves (39).

(b) Let, on the other hand, $(v, w) \in H(\text{curl}, D) \times V_k$ be a solution of (38) and (39). Again we set $u = v - w$ in $D$. But this time we extend $u$ by the extension of $A_k v + B_k v - A_k \hat{f}$ to $\mathbb{R}^3 \setminus D$. Then $u$ satisfies the radiation condition and, by Lemma 2.2 again,

$$
\iiint_{\mathbb{R}^3} [\text{curl} u \cdot \text{curl} \psi - k^2 u \cdot \psi] \, dx = \iint_D [k^2 p (v - \hat{f}) \cdot \psi + q \text{curl} v \cdot \text{curl} \psi] \, dx
$$

for all $\psi \in H(\text{curl}, \mathbb{R}^3)$ with compact support. We choose two open balls $B$ and $\hat{B}$ such that $\overline{D} \subset B$ and $\overline{\mathcal{B}} \subset B$ and a scalar function $\phi \in C^\infty(\mathbb{R}^3)$ with compact support in $B$ and $\phi \equiv 1$ in $B$. For some $a \in C^2(\overline{\mathcal{B}} \setminus D, \mathbb{C}^3)$ we define

$$
\tilde{\psi}(x) = (k^2 + \nabla \text{div}) \iint_{\overline{\mathcal{B}} \setminus D} a(y) \Phi(x, y) \, dy, \quad x \in \mathbb{R}^3,
$$

and set $\psi = \phi \tilde{\psi}$. Then $\psi \in H(\text{curl}, \mathbb{R}^3)$ has support in $\hat{B}$, $\text{curl}^2 \psi - k^2 \psi = 0$ in $D$, $\text{curl}^2 \psi - k^2 \psi = k^2 a$ in $B \setminus \overline{D}$ and $\nu \times \psi$ and $\nu \times \text{curl} \psi$ are continuous on $\partial B \cup \partial D$ (and vanish on $\partial \hat{B}$). Therefore, substituting this form of $\psi$ into (40) yields

$$
\iint_{\hat{B}} [\text{curl} u \cdot \text{curl} \psi - k^2 u \cdot \psi] \, dx = 0
$$
because of $\psi_D \in V_{k,0}$ and (39). Application of Green’s first theorem in $\hat{B}$ yields
\[
0 = \int_B u \cdot [\text{curl}^2 \psi - k^2 \psi] \, dx = k^2 \int_{\hat{B} \setminus D} u \cdot a \, dx + \int_{\hat{B} \setminus B} u \cdot (\text{curl}^2 \psi - k^2 \psi) \, dx.
\]
Since $\psi$ and $u$ are smooth in $\hat{B} \setminus B$ and $u$ satisfies $\text{curl}^2 u - k^2 u = 0$ in the exterior of $D$ we can apply Green’s second theorem in $\hat{B} \setminus B$ which yields
\[
0 = k^2 \int_{\hat{B} \setminus D} u \cdot a \, dx + \int_{\partial B} [u \cdot (\nu \times \text{curl} \hat{\psi}) - \hat{\psi} \cdot (\nu \times \text{curl} u)] \, ds.
\]
The boundary integral vanishes since $u$ and $\hat{\psi}$ satisfy both the Silver-Müller radiation condition. Since this holds for all such vector fields $a$ we conclude that $u$ vanishes in $\hat{B} \setminus D$. Therefore, from Theorem 2.3 we conclude that $(v, w)$ solves (36), (37).

It is convenient to identify $H(\text{curl}, D)$ with a subspace of $X = L^2(D, \mathbb{C}^3) \times L^2(D, \mathbb{C}^3)$. We set $v_1 = v$ and $v_2 = \text{curl} v$ and write (38) and the curl of this equation as the following system in $X$:
\[
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} K^{(1)}_k \\ K^{(2)}_k \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w \\ \text{curl} w \end{pmatrix} - \begin{pmatrix} F_1 \\ F_2 \end{pmatrix},
\]
where
\[
(C_1 \psi)(x) = \nabla \text{div} \int_D p(y) \psi(y) \Phi_0(x, y) \, dy \quad x \in D,
\]
\[
(C_2 \psi)(x) = \text{curl}^2 \int_D q(y) \psi(y) \Phi_0(x, y) \, dy \quad x \in D,
\]
\[
(K^{(1)}_k \psi)(x) = k^2 \int_D p(y) \psi(y) \Phi_k(x, y) \, dy + \nabla \text{div} \int_D p(y) \psi(y) [\Phi_k(x, y) - \Phi_0(x, y)] \, dy \quad x \in D,
\]
\[
(K^{(2)}_k \psi)(x) = \text{curl} \int_D q(y) \psi(y) \Phi_k(x, y) \, dy \quad x \in D,
\]
\[
(K^{(3)}_k \psi)(x) = k^2 \text{curl} \int_D p(y) \psi(y) \Phi_k(x, y) \, dy \quad x \in D,
\]
\[
(K^{(4)}_k \psi)(x) = \text{curl}^2 \int_D q(y) \psi(y) [\Phi_k(x, y) - \Phi_0(x, y)] \, dy \quad x \in D,
\]
and
\[
F_1(x) = (I + \frac{1}{k^2} \nabla \text{div}) \int_D f(y) \Phi_k(x, y) \, dy, \quad x \in D,
\]
\[
F_2(x) = \text{curl} F_1(x) = \text{curl} \int_D f(y) \Phi_k(x, y) \, dy, \quad x \in D.
\]
In order to avoid the matrix - vector forms we agree to abbreviate vectors (of vector fields) by using the tilde sign, i.e. 
\( \tilde{v} = (v_1, v_2) \) and analogously for \( \tilde{\psi} \), \( \tilde{F} \), etc. Furthermore, we set 
\( \tilde{w} = (w^1, w^2) \) and
\[
\tilde{C} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \quad \text{and} \quad \tilde{K}_k = \begin{pmatrix} K_k^{(1)} & K_k^{(2)} \\ K_k^{(3)} & K_k^{(4)} \end{pmatrix}.
\]

Then (41) takes the form
\[
\tilde{v} - \tilde{C}\tilde{v} - \tilde{K}_k\tilde{v} = \tilde{w} - \tilde{F}.
\]

We equip \( X = L^2(D, \mathbb{C}^3) \times L^2(D, \mathbb{C}^3) \) with the inner product
\[
\langle \tilde{v}, \tilde{\psi} \rangle_k = \int_D \left[ k^2 p \, v_1 \cdot \psi_1 + q \, v_2 \cdot \psi_2 \right] \, dx, \quad \tilde{v}, \tilde{\psi} \in X,
\]
and note that the corresponding norm \( \| \cdot \|_k \) – which depends on \( k \) – is equivalent to the standard norm of \( L^2(D, \mathbb{C}^3) \times L^2(D, \mathbb{C}^3) \).

All of the components of \( \tilde{K}_k \) are compact operators from \( L^2(D, \mathbb{C}^3) \) into itself due to the weak singularity of the kernel functions. Therefore, \( \tilde{K}_k \) is compact from \( X \) into itself. The operator \( \tilde{C} \) is a contraction operator in \( X \) with respect to \( \| \cdot \|_k \). This is seen by using standard methods of Fourier analysis. Indeed, the operators \( W_1 \) and \( W_2 \), defined (formally) by
\[
(W_1\psi)(x) = \nabla \text{div} \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \psi(y) \, dy, \quad x \in \mathbb{R}^3,
\]
\[
(W_2\psi)(x) = \text{curl}^2 \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \psi(y) \, dy, \quad x \in \mathbb{R}^3,
\]
are transformed into the Fourier space as
\[
\mathcal{F}(W_1\psi)(\xi) = -\xi \tilde{\xi}^\top \mathcal{F}(\psi)(\xi),
\]
\[
\mathcal{F}(W_2\psi)(\xi) = [I - \xi \tilde{\xi}^\top] \mathcal{F}(\psi)(\xi),
\]
where \( \tilde{\xi} = \xi/|\xi| \), and the Fourier transform \( \mathcal{F} \) is given as the extension of
\[
(\mathcal{F}\psi)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\xi \cdot x} \psi(x) \, dx, \quad \xi \in \mathbb{R}^3,
\]
to tempered distributions. The unitarity of the Fourier transform yields
\[
\|W_\ell\psi\|_{L^2(\mathbb{R}^3)} = \|\mathcal{F}(W_\ell\psi)\|_{L^2(\mathbb{R}^3)} = \|\mathcal{F}\psi\|_{L^2(\mathbb{R}^3)} = \|\psi\|_{L^2(\mathbb{R}^3)}
\]
for \( \ell = 1, 2 \) where we have also used that \( \varphi \mapsto [I - \tilde{\xi} \tilde{\xi}^\top] \varphi \) and \( \varphi \mapsto -\tilde{\xi} \tilde{\xi}^\top \varphi \) are projection operators of norm one. From \( C_1\psi = W_1(p\psi)|_D \) and \( C_2\psi = W_2(q\psi)|_D \) we conclude that
\[
\|\mathcal{C}\psi\|^2_k = k^2 \|\sqrt{p} C_1\psi\|^2_{L^2(D)} + \|\sqrt{q} C_2\psi\|^2_{L^2(D)} = k^2 \|p\psi\|^2_{L^2(\mathbb{R}^3)} + \|q\psi\|^2_{L^2(\mathbb{R}^3)} \leq k^2 \|p\|^2_{L^\infty} \|\psi\|^2_{L^2(\mathbb{R}^3)} + \|q\|^2_{L^\infty} \|\psi\|^2_{L^2(\mathbb{R}^3)} \leq \max\{\|p\|^2_{L^\infty}, \|q\|^2_{L^\infty}\} \|\psi\|^2_k.
\]
From assumption (3.3) we have that $\|p\|_\infty < 1$ and $\|q\|_\infty < 1$ and thus

$$\|\tilde{\psi}\|_k \leq \alpha \|\psi\|_k, \quad \psi \in X,$$

where $\alpha = \max\{\|p\|_\infty, \|q\|_\infty\} < 1$.

In this decomposition the subspace $V_k$ corresponds to

$$\tilde{V}_k = \left\{ \tilde{v} \in X : \iint_D [v_2 \cdot \psi_1 - v_1 \cdot \psi_2] dx = 0 \right\}$$

which is the weak form of

$$\tilde{v} = \tilde{\psi}.$$

We formulate Lemma 3.4 in this new setting as a corollary.

**Corollary 3.5.** Let $f \in L^2(D, \mathbb{C}^3)$ and define $\tilde{C}, \tilde{K}_k,$ and $\tilde{F}$ as before. Then $(v, w) \in H(\text{curl}, D) \times \tilde{V}_k$ solves (34), (35) for $g = h = 0$ if, and only if, $(\tilde{v}, \tilde{w}) \in X \times \tilde{V}_k$ satisfies

$$\tilde{v} - \tilde{C} \tilde{v} - \tilde{K}_k \tilde{v} = \tilde{w} - \tilde{F}$$

and

$$\langle \tilde{v}, \tilde{\psi} \rangle_k = \iint_D f \cdot \bar{\psi}_1 dx \quad \text{for all } \tilde{\psi} \in \tilde{V}_{k,0}.$$

**3.1. THE INTERIOR TRANSMISSION EIGENVALUE PROBLEM.** Now we consider the interior transmission eigenvalue problem, i.e. we set $f \equiv 0$ in (34), (36), (38), and (39) and $\tilde{F} \equiv 0$ in (41) and (44) and, of course, $g = h = 0$. Similarly to the approach in [5] or [7] for the scalar case (see also [10] for the anisotropic case) we will project equation (44) for $\tilde{F} = 0$ onto the orthogonal complement

$$\tilde{V}_{k,0} = \left\{ \tilde{v} \in X : \iint_D [k^2 p v_1 \cdot \bar{\psi}_1 + q v_2 \cdot \bar{\psi}_2] dx = 0 \right\}$$

of $\tilde{V}_{k,0}$.

From (39) for $f = 0$ we observe that $\tilde{v} \in \tilde{V}_{k,0}$. Therefore, if $P_k$ is the orthogonal projector from $X$ onto $\tilde{V}_{k,0}$ and and $(\tilde{v}, \tilde{w}) \in X \times \tilde{V}_{k,0}$ solves (44) for $\tilde{F} = 0$ then $\tilde{v}$ solves

$$\tilde{v} - P_k \tilde{C} \tilde{v} - P_k \tilde{K}_k \tilde{v} = 0.$$

And vice versa: If $\tilde{v} \in \tilde{V}_{k,0}$ solves (47) then $(\tilde{v}, \tilde{w}) \in X \times \tilde{V}_{k,0}$ solves (44) for $\tilde{F} = 0$ where $\tilde{w} = \tilde{v} - \tilde{C} \tilde{v} - \tilde{K}_k \tilde{v} = (P_k - I)(\tilde{C} \tilde{v} + \tilde{K}_k \tilde{v})$.

Before we apply the analytic Fredholm theory we show that $P_k$ and $\tilde{K}_k$ depend analytically on $k$. For $\tilde{K}_k$ this is obvious since the kernels depend analytically on $k$. It remains to study the projection operator $P_k$. We will first derive an explicit expression of $P_k$.

In the following, we denote by $[H_0(\text{curl}, D)]^*$ the space of bounded conjugate-linear functionals on $H_0(\text{curl}, D)$ and by $\langle w, \psi \rangle_*$ the dual form in the dual system

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3For the notions of weakly analytic and strongly analytic operator valued functions we refer to, e.g., [7].
\langle [H_0(\text{curl}, D)]^*, H_0(\text{curl}, D) \rangle. \) We set \( Y = H_0(\text{curl}, D) \times H_0(\text{curl}, D) \) and define the operator \( T_k : X \rightarrow Y^* \) by

\[
(48) \quad \langle T_k \tilde{v}, \tilde{\psi} \rangle_* = \int_D [v_1 \cdot \text{curl} \tilde{\psi}_1 - v_2 \cdot \tilde{\psi}_1] \, dx + \int_D [k^2 v_1 \cdot \tilde{\psi}_2 - v_2 \cdot \text{curl} \tilde{\psi}_2] \, dx
\]

for \( \psi_1, \psi_2 \in H_0(\text{curl}, D) \) and \( v_1, v_2 \in L^2(D, \mathbb{C}^3) \). Obviously, \( T_k \) is linear and bounded and \( \tilde{V}_{k,0} = N(T_k) \). The adjoint operator \( T_k^* : Y \rightarrow X \) with respect to the inner product \( \langle \cdot, \cdot \rangle_k \) is given by the Riesz-representation, i.e. \( T_k^* \tilde{\psi} = \tilde{w} \) where

\[
(49) \quad w_1 = \frac{1}{k^2 p} (k^2 \psi_2 + \text{curl} \psi_1), \quad w_2 = -\frac{1}{q} (\psi_1 + \text{curl} \psi_2)
\]

because then \( \langle T_k \tilde{v}, \tilde{\psi} \rangle_* = \langle \tilde{v}, \tilde{w} \rangle_k \).

**Theorem 3.6.** The orthogonal projection operator \( P_k \) from \( X \) onto \( \tilde{V}_{k,0} \) with respect to the inner product \( \langle \cdot, \cdot \rangle_k \) is given by

\[
(50) \quad P_k = T_k^*(T_k T_k^*)^{-1} T_k.
\]

where \( T_k \) and \( T_k^* \) are defined by (48) and (49), respectively. In particular, the operator \( T_k T_k^* \) is an isomorphism from \( Y = H_0(\text{curl}, D) \times H_0(\text{curl}, D) \) onto \( Y^* = [H_0(\text{curl}, D)]^* \times [H_0(\text{curl}, D)]^* \).

**Proof.** The main part consists of showing the existence of \( c > 0 \) with

\[
(51) \quad \|T_k^* \tilde{v}\|_k \geq c \|\tilde{v}\|_Y \quad \text{for all } \tilde{v} \in Y = H_0(\text{curl}, D) \times H_0(\text{curl}, D).
\]

Assume, on the contrary, that such an estimate fails to hold. Then there exists a sequence \( (\tilde{v}^{(j)}) \) in \( Y \) with \( \|\tilde{v}^{(j)}\|_Y = 1 \) and \( \|T_k^* \tilde{v}^{(j)}\|_k \rightarrow 0 \) as \( j \) tends to infinity, i.e.

\[
(52) \quad v_2^{(j)} + \frac{1}{k^2} \text{curl} v_1^{(j)} \rightarrow 0, \quad \psi_1^{(j)} + \text{curl} v_2^{(j)} \rightarrow 0 \quad \text{in } L^2(D, \mathbb{C}^3).
\]

The boundedness of \( \|\tilde{v}^{(j)}\|_Y \) yields the existence of a weakly convergent subsequence, again denoted by \( (\tilde{v}^{(j)}) \), such that \( v_\ell^{(j)} \rightharpoonup v_\ell \) weakly in \( H(\text{curl}, D) \) as \( j \) tends to infinity for some \( v_\ell \in H_0(\text{curl}, D) \), \( \ell = 1, 2 \). Letting \( j \) tend to infinity in (52) yields that \( v_1, v_2 \) satisfy

\[
(53) \quad v_2 + \frac{1}{k^2} \text{curl} v_1 = 0, \quad v_1 + \text{curl} v_2 = 0 \quad \text{in } D.
\]

Therefore, \( v_1 \) satisfies \( k^2 v_1 - k^2 v_1 = 0 \) in \( D \) and \( v_1, \text{curl} v_1 \in H_0(\text{curl}, D) \). By extending \( v_1 \) by zero outside of \( D \) one observes that \( \text{curl}^2 v_1 - k^2 v_1 = 0 \) in all of \( \mathbb{R}^3 \). The unique continuation property yields \( v_1 \equiv v_2 \equiv 0 \) in \( D \). Now we make use of the Helmholtz decomposition. We write \( v_\ell^{(j)} \) in the forms

\[
v_\ell^{(j)} = \tilde{v}_\ell^{(j)} + \text{grad} \phi_\ell^{(j)}, \quad j \in \mathbb{N}, \quad \ell = 1, 2,
\]

with \( \phi_\ell^{(j)} \in H^3(D) \) normalized to \( \int_D \phi_\ell^{(j)} \, dx = 0 \) and \( \tilde{v}_\ell^{(j)} \in H(\text{curl}, D) \) such that \( \text{div} \tilde{v}_\ell^{(j)} = 0 \) in \( D \) and \( \nu \cdot \tilde{v}_\ell^{(j)} = 0 \) on \( \partial D \). This has again to be understood in the variational sense, i.e. \( \int_D \tilde{v}_\ell^{(j)} \cdot \nabla \varphi \, dx = 0 \) for all \( \varphi \in H^1(D) \). Since this decomposition is orthogonal it is easily seen that also \( \tilde{v}_\ell^{(j)} \) and \( \text{grad} \phi_\ell^{(j)} \) converge weakly to zero in \( H_0(\text{curl}, D) \) and \( H^1(D) \), respectively.
Since \( \|v^{(j)}_\ell\|_{H(\text{curl},D)} \) is bounded, \( \text{div} v^{(j)}_\ell = 0 \) in \( D \) and \( \nu \cdot v^{(j)}_\ell = 0 \) on \( \partial D \) a well known compactness result (cf. [24]) yields that \( v^{(j)}_\ell \) converge to zero in the norm of \( L^2(D, \mathbb{C}^3) \). From this and (52) we conclude that
\[
(53) \quad \nabla \phi_2^{(j)} + \frac{1}{k^2} \text{curl} v_1^{(j)} \longrightarrow 0 \quad \text{and} \quad \nabla \phi_1^{(j)} + \text{curl} v_2^{(j)} \longrightarrow 0 \quad \text{in} \quad L^2(D, \mathbb{C}^3).
\]
We multiply the first equation by \( \nabla \phi_2^{(j)} \) and integrate over \( D \). This yields, since \( \nabla \phi_2^{(j)} \) is bounded in \( L^2(D, \mathbb{C}^3) \),
\[
\iint_D |\nabla \phi_2^{(j)}|^2 \, dx + \frac{1}{k^2} \iint_D \text{curl} v_1^{(j)} \cdot \nabla \phi_2^{(j)} \, dx \longrightarrow 0.
\]
The second integral vanishes by Green’s theorem since \( \nu \times v^{(j)}_1 = 0 \) on \( \partial D \). Therefore, \( \phi_2^{(j)} \) tends to zero in \( H^1(D) \). By the same argument also \( \phi_1^{(j)} \) tends to zero in \( H^1(D) \) and thus also \( \text{curl} v^{(j)}_1 \to 0 \) in \( L^2(D, \mathbb{C}^3) \) for \( \ell = 1, 2 \). This, finally, contradicts \( \|\tilde{v}_\ell\|_{H(\text{curl},D)} = 1 \) and proves estimate (51).

From (51) we observe that \( T_k T_k^* \) is coercive, i.e.
\[
\langle T_k T_k^* \tilde{\psi}, \tilde{\psi} \rangle \geq c^2 \|	ilde{\psi}\|^2_{\mathcal{Y}}
\]
which shows that \( T_k T_k^* \) is an isomorphism from \( \mathcal{Y} \) onto \( \mathcal{Y}^* \). Therefore, \( P_k \) is well defined.

Finally, we see directly that \( P_k \tilde{\psi} \in \mathcal{R}(T_k^*) \subset \mathcal{N}(T_k)^\perp = \tilde{\mathcal{V}}_{k,0} \) and, furthermore, \( T_k(P_k \tilde{\psi} - \tilde{\psi}) = (T_k T_k^*)(T_k T_k^*)^{-1} T_k \tilde{\psi} - T_k \tilde{\psi} = 0 \), i.e. \( P_k \tilde{\psi} - \tilde{\psi} \in \mathcal{N}(T_k) = \tilde{\mathcal{V}}_{k,0} \). This ends the proof.

From the explicit form (48) we observe that \( k \mapsto T_k \) is weakly analytic and hence strongly analytic in some neighborhood \( U \subset \mathbb{C} \) of the positive real semi-axis where we consider \( X \) to be equipped with the standard inner product (which is independent of \( k \)). The same holds for the operator \( k \mapsto T_k^* \) given by (49). Therefore, \( k \mapsto P_k \) is strongly analytic.

We go back to (47) and note that also \( \|P_k \tilde{C}\|_k \leq \alpha \) with \( \alpha = \max\{\|p\|_{\infty}, \|q\|_{\infty}\} < 1 \) since \( P_k \) has norm one as an orthogonal projection operator. Therefore, \( I - P_k \tilde{C} \) is boundedly invertible in \( X \). Furthermore, \( k \mapsto I - P_k \tilde{C} \) is analytic where we consider again \( X \) to be equipped with the standard inner product. Thus also \( k \mapsto (I - P_k \tilde{C})^{-1} \) is analytic, and we rewrite (47) as
\[
(54) \quad \tilde{v} - (I - P_k \tilde{C})^{-1} P_k \tilde{K}_k \tilde{v} = 0.
\]
Summarizing, we have shown that \( (I - P_k \tilde{C})^{-1} P_k \tilde{K}_k \) is compact and depends analytically on \( k \) in some neighborhood \( U \subset \mathbb{C} \) of \( \mathbb{R}_{>0} \). We are now in the situation where we can apply the analytic Fredholm theory, see [7], Theorem 8.19.

**Theorem 3.7.** Let Assumption 3.3 hold. Then the set of interior transmission eigenvalues is discrete.

**Proof.** It remains to show that there exists some (sufficiently small) \( k > 0 \) which is not an eigenvalue. First we note that from the definitions of the operators \( K^{(\ell)}_k \) for \( \ell = 1, 2, 3, 4 \) there follows the existence of \( \hat{c} > 0 \) (independent of \( k \)) such that
\[
\|K^{(\ell)}_k \psi\|_{L^2(D)} \leq \hat{c} k^2 \|p\psi\|_{L^2(D)}, \quad \ell = 1, 3,
\]
and
\[ \|K^{(2)}_k \psi\|_{L^2(D)} \leq \hat{c} \|q\psi\|_{L^2(D)}, \quad \|K^{(4)}_k \psi\|_{L^2(D)} \leq \tilde{c} k^2 \|q\psi\|_{L^2(D)} \]
for all \( k > 0 \) and \( \psi \in L^2(D, \mathbb{C}^3) \). Furthermore, from the proof of (43) we recall that
\[ \|C_1 \psi\|_{L^2(D)} = \|W_1(p\psi)|D\|_{L^2(D)} \leq \|p\psi\|_{L^2(D)} \]
and, analogously,
\[ \|C_2 \psi\|_{L^2(D)} \leq \|q\psi\|_{L^2(D)} \].
We choose \( k > 0 \) so small such that
\[ (1 + \hat{c} k^2 + \tilde{c} k) \|p\|_\infty < 1 \quad \text{and} \quad (1 + \hat{c} k^2 + \tilde{c} k) \|q\|_\infty < 1 \]
which is possible by Assumption 3.3. Let now \( \tilde{v} = \left( \begin{array}{c} \psi \\ v \end{array} \right) \) satisfy (47). Then, since \( \tilde{v} \in \tilde{V}_{k,0} \) and \( P_k \) is an orthogonal projector,
\[ 0 = \langle \tilde{v} - P_k \hat{C} \tilde{v} - P_k \hat{K}_k \tilde{v}, \tilde{v} \rangle_k = \| \tilde{v} \|_k^2 - \langle \hat{C} \tilde{v} + \hat{K}_k \tilde{v}, \tilde{v} \rangle_k \]
and thus by the definitions of \( \| \cdot \|_k \) and \( \hat{C} \) and \( \hat{K}_k \)
\[ k^2 \sqrt{p} |v_1|^2_{L^2(D)} + \| \sqrt{q} v_2 \|^2_{L^2(D)} = \]
\[ = k^2 \langle C_1 v_1 + K^{(1)}_k v_1 + K^{(2)}_k v_2, p v_1 \rangle_{L^2(D)} + \langle C_2 v_2 + K^{(3)}_k v_1 + K^{(4)}_k v_2, q v_2 \rangle_{L^2(D)} \].
We set \( \hat{v}_2 = \frac{1}{k} v_2 \) and divide the previous equation by \( k^2 \). This yields
\[ \| \sqrt{p} v_1 \|^2_{L^2(D)} + \| \sqrt{q} v_2 \|^2_{L^2(D)} = \]
\[ = \langle C_1 v_1 + K^{(1)}_k v_1 + K^{(2)}_k v_2, p v_1 \rangle_{L^2(D)} + \langle C_2 v_2 + \frac{1}{k} K^{(3)}_k v_1 + K^{(4)}_k v_2, q v_2 \rangle_{L^2(D)} \]
\[ \leq \| p v_1 \|^2_{L^2(D)} + \hat{c} k^2 \| p v_1 \|^2_{L^2(D)} + \tilde{c} k \| q \hat{v}_2 \|^2_{L^2(D)} \| \hat{v}_2 \|_{L^2(D)} + \| q v_2 \|^2_{L^2(D)} \]
\[ \leq (1 + \hat{c} k^2 + \tilde{c} k) \left[ \| p v_1 \|^2_{L^2(D)} + \| q \hat{v}_2 \|^2_{L^2(D)} \right] \]
\[ \leq (1 + \hat{c} k^2 + \tilde{c} k) \left[ \| p \|_\infty \| \sqrt{p} v_1 \|^2_{L^2(D)} + \| q \|_\infty \| \sqrt{q} v_2 \|^2_{L^2(D)} \right] \]
where we have used the binomial inequality \( 2ab \leq a^2 + b^2 \). Therefore,
\[ 1 - (1 + \hat{c} k^2 + \tilde{c} k) \|p\|_\infty \| \sqrt{p} v_1 \|^2_{L^2(D)} + \left[ 1 - (1 + \hat{c} k^2 + \tilde{c} k) \|q\|_\infty \right] \| \sqrt{q} \hat{v}_2 \|^2_{L^2(D)} \leq 0 \]
which, together with (55), implies that \( v_1 \) and \( \hat{v}_2 \) have to vanish.
\[ \square \]

3.2. The Interior Transmission Boundary Value Problem. In this subsection we consider (34), (35) for given \( f \in L^2(D, \mathbb{C}^3) \) (but still \( g = h = 0 \)). Throughout this section we assume that \( k^2 \) is not an interior transmission eigenvalue in the sense of Definition 3.1. Therefore, by Corollary 3.5, the only solution \((\tilde{v}, \tilde{w}) \in \tilde{V}_{k,0}^{+} \times \tilde{V}_{k,0}^{-}\)
of the homogeneous equation
\[ \tilde{v} - \hat{C} \tilde{v} - \hat{K}_k \tilde{v} = \tilde{w} \]
is the trivial one \( (\tilde{v}, \tilde{w}) = (0, 0) \).

By Corollary 3.5 again, problem (34), (35) (for \( g = h = 0 \)) is equivalent to solving the equations (44) and (45). We choose any fixed solution \( \tilde{w}_{inh} \in \tilde{V}_{k} \) and replace \( \tilde{w} \) by \( \tilde{w} + \tilde{w}_{inh} \). Therefore, we have to determine \((\tilde{v}, \tilde{w}) \in X \times \tilde{V}_{k,0}^{-}\) which satisfies
\[ \tilde{v} - \hat{C} \tilde{v} - \hat{K}_k \tilde{v} = \tilde{w}_{inh} + \tilde{w} - \hat{F} \]
and
\[ \langle \tilde{v}, \tilde{w} \rangle_k = \int_{D} f \cdot \tilde{w} \, dx \quad \text{for all} \ \tilde{w} \in \tilde{V}_{k,0}^{-}. \]
We note that the integral on the right hand side of (58) defines a linear and bounded functional on X. Therefore, by the representation theorem of Riesz there exists \( \tilde{f} \in X \) such that \( \langle \tilde{f}, \psi \rangle_k = \int_D f \cdot \psi \, dx \) for all \( \psi \in X \). Therefore, we have to determine \( (\tilde{\nu}, \tilde{\omega}) \in X \times \tilde{V}_{k,0} \) with (57) and \( \tilde{\nu} - \tilde{f} \in \tilde{V}_{k,0} \). Setting \( \tilde{u} = \tilde{\nu} - \tilde{f} \) this equation is equivalent to determine \( (\tilde{u}, \tilde{w}) \in \tilde{V}_{k,0}^+ \times \tilde{V}_{k,0} \) with

\[
\tilde{u} - \tilde{C} \tilde{u} - \tilde{K}_k \tilde{u} = \tilde{w} + \tilde{G}
\]

where \( \tilde{G} = \tilde{w}_{\text{inh}} - \tilde{F} - \tilde{f} + \tilde{C} \tilde{f} + \tilde{K}_k \tilde{f} \). We determine \( \tilde{u} \in \tilde{V}_{k,0}^+ \) such that

\[
\tilde{u} - P_k \tilde{C} \tilde{u} - P_k \tilde{K}_k \tilde{u} = P_k \tilde{G}.
\]

This is uniquely possible by the Riesz-Fredholm theory since \( P_k \tilde{K}_k \) is compact and \( P_k \tilde{C} \) is a contraction in \( X \) and the homogeneous equation

\[
\tilde{u} - P_k \tilde{C} \tilde{u} - P_k \tilde{K}_k \tilde{u} = 0
\]

admits only the trivial solution \( \tilde{u} = 0 \) in \( \tilde{V}_{k,0}^+ \) by the equivalence of (47) and (44) for \( \tilde{F} = 0 \). Repeating this argument, we set \( \tilde{w} = \tilde{u} - \tilde{C} \tilde{u} - \tilde{K}_k \tilde{u} - \tilde{G} = (P_k - I)(\tilde{C} \tilde{u} + \tilde{K}_k \tilde{u} + \tilde{G}) \). From this we observe that \( \tilde{w} \in \tilde{V}_{k,0} \), and \( (\tilde{u}, \tilde{w}) \in \tilde{V}_{k,0}^+ \times \tilde{V}_{k,0} \) solves (59).

We have finally shown:

**Theorem 3.8.** Let Assumption 3.3 hold and assume that \( k^2 \) is not an interior transmission eigenvalue. Then, for every \( f \in L^2(D, \mathbb{C}^3) \) and tangential vector fields \( g \in H^{3/2}(\partial D, \mathbb{C}^3) \), and \( h \in H^{1/2}(\partial D, \mathbb{C}^3) \) there exists a unique solution \( (v, w) \) in \( H(\text{curl}, D) \times H(\text{curl}, D) \) of (34), (35).

**References**


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