Research Article

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The interior transmission eigenvalue problem for a spherically-symmetric domain with anisotropic medium and a cavity

Abstract: We consider the scattering of spherically-symmetric acoustic waves by an anisotropic medium and a cavity. While there is a large number of recent works devoted to the scattering problems with cavities, existence of an infinite set of transmission eigenvalues is an open problem in general. In this paper we prove existence of an infinite set of transmission eigenvalues for anisotropic Helmholtz and Schrödinger equations in a spherically-symmetric domain with a cavity. Further in this paper we consider the corresponding inverse problem. Under some assumptions we prove the uniqueness in the inverse problem.

Keywords: Helmholtz equation, transmission eigenvalues, Liouville transform, asymptotics of eigenvalues

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1 The direct problem

We consider the following problem

\[ \nabla \cdot \sigma \nabla f + \lambda f = 0 \quad \text{in} \quad D_1 \setminus \overline{D_0}, \quad (1.1) \]
\[ \Delta f + \lambda f = 0 \quad \text{in} \quad D_1, \quad (1.2) \]
\[ w = \nu \quad \text{on} \quad \partial D_1, \quad (1.3) \]
\[ \nu \cdot \nabla w = \frac{\partial w}{\partial \nu} \quad \text{on} \quad \partial D_1, \quad (1.4) \]
\[ w = 0 \quad \text{on} \quad \partial D_0. \quad (1.5) \]

where \( D_1 = B(0, r_1) \) is the open ball with the center at the origin of \( \mathbb{R}^3 \) and radius \( r_1 \), \( D_0 = B(0, r_0) \) \( (r_0 < r_1) \) is a concentric ball with a smaller radius, \( (w, \nu) \) is the pair of fields, \( \nabla \) denotes the gradient operator, \( \Delta \) denotes the Laplacian, \( \cdot \) is the formal inner product in \( \mathbb{R}^3 \), \( \nu \) represents the outward unit normal to \( \partial D_1 \), \( \lambda \) is the spectral parameter, \( \rho : D_1 \setminus \overline{D_0} \to \mathbb{C} \) is a function corresponding to the refractive index of the medium at location \( x \), \( a : D_1 \setminus \overline{D_0} \to \mathbb{R} \) is a given function corresponding to the anisotropy of the medium. At the beginning we will assume that \( \rho \in L^\infty(D_1 \setminus \overline{D_0}), a \in C^1(D_1 \setminus \overline{D_0}) \) and

\[ \text{Im} \rho(x) \geq 0 \quad \text{(a.e. on} \ D_1 \setminus \overline{D_0}), \quad (1.6) \]
\[ a(x) \geq a_0 \quad (x \in D_1 \setminus \overline{D_0}) \]

with some positive constant \( a_0 \).

The problem (1.1)–(1.5) was first investigated in [3]. Let us bring some definitions from there. Define

\[ H_\lambda^1(D_1 \setminus \overline{D_0}) = \{ u \in H^1(D_1 \setminus \overline{D_0}) : \Delta u \in L^2(D_1 \setminus \overline{D_0}) \}. \]

Definition 1.1. A weak solution to (1.1)–(1.5) is a pair of functions \( (w, \nu) \in H^1(D_1 \setminus \overline{D_0}) \times H^1(D_1) \) satisfying (1.1), (1.2) in the distributional sense such that \( w = 0 \) on \( \partial D_0 \), \( w - \nu \in H_\lambda^1(D_1 \setminus \overline{D_0}) \) and \( w - \nu = 0, \nu \cdot \nabla w - \frac{\partial w}{\partial \nu} = 0 \) on \( \partial D_1 \).
**Definition 1.2.** The values of \( \lambda \geq 0 \) for which the problem (1.1)–(1.5) has a nontrivial solution are called the transmission eigenvalues.

For the case without obstacles (i.e. when the problem (1.1)–(1.5) is considered with \( D_0 = \emptyset \)) and with constant coefficients \( a, \rho \) the transmission eigenvalue problem was first considered in [8].

![Diagram of the problem](image)

**Theorem 1.3.** If \( \int_{D_1} a \| \nabla w \|^2 dx \) then there are no positive transmission eigenvalues for which the corresponding solution \( \omega \) belongs to the space \( H^2(D_1 \setminus D_0) \).

**Proof.** Let \( (w, \psi) \in H \) be a solution of (1.1)–(1.5) with \( w \in H^2(D_1 \setminus D_0) \) and \( \lambda > 0 \). Taking \( w = w^* = \psi \) from (1.7) we get

\[
\int_{D_1 \setminus D_0} a|\nabla w|^2 dx - \int_{D_1} |\nabla \psi|^2 dx + \lambda \int_{D_1} |\psi|^2 dx - \lambda \int_{D_1 \setminus D_0} \rho |w|^2 dx = 0,
\]

that is,

\[
\int_{D_1 \setminus D_0} \rho |w|^2 dx = \frac{1}{\lambda} \int_{D_1 \setminus D_0} a|\nabla w|^2 dx - \frac{1}{\lambda} \int_{D_1} |\nabla \psi|^2 dx + \int_{D_1} |\psi|^2 dx.
\]

Taking the imaginary part from both sides leads to the equality

\[
\int_{D_1 \setminus D_0} |w|^2 \text{Im} \rho \, dx = 0.
\]
From here in view of (1.6) we get
\[ |w|^2 \Im \rho = 0 \quad \text{(a.e. on } D_1 \setminus \overline{D}_0). \]
This equality together with (1.8) shows that \( w \) vanishes on \( G \). To complete the proof now we will show that the unique continuation principle holds for the solution \( w \).

Equation (1.1) can be rewritten as
\[ \Delta w = -\frac{V_a}{a} \cdot \nabla w - \frac{\lambda \rho}{a} w, \]
hence in \( D_1 \setminus \overline{D}_0 \)
\[ |\Delta w| \leq C(|\nabla w| + |w|) \]
with some positive constant \( C \). Lemma 8.5 from [4, p. 273] states that if a function from \( H^2(D_1 \setminus \overline{D}_0) \) satisfies this kind of estimate and vanishes in an open subset of \( D_1 \setminus \overline{D}_0 \), then that function is identically zero. Therefore we get \( w \equiv 0 \). Now the uniqueness theorem for the Dirichlet problem (1.2), (1.3) implies that \( \nu \equiv 0 \). Thus, we get \( (w, \nu) = (0, 0) \), hence \( \lambda \) is not a transmission eigenvalue, and the proof is complete. \( \square \)

In view of Theorem 1.3 it is natural to assume that the coefficient \( \rho \) is a real-valued function.

In the following we will consider the spherically-symmetric case. More precisely, we assume that
\[ \rho = \rho(\|x\|) = \rho(r), \quad a = a(\|x\|) = a(r) \]
and corresponding wave functions \((w, \nu)\) are spherically-symmetric and continuous in \((\overline{D}_1 \setminus D_0) \times \overline{D}_1\). It is proved in [3, Theorems 3.3, 3.5, 3.10] that under some assumptions the set of transmission eigenvalues is discrete and there exists at least one transmission eigenvalue. For the spherically-symmetric transmission eigenvalue problem without inclusion with \( a \equiv 1 \) these facts were proved earlier in [5]. Our first result states the existence of an infinite set of transmission eigenvalues for the problem (1.1)–(1.5) (see Theorem 1.5 of this paragraph). For the case without inclusion this result is proved in [4, Theorem 8.13, p. 281]. In Section 2 we will investigate the corresponding inverse problem for (1.1)–(1.5). Such kinds of investigations for the case without obstacles are done in [1, 12].

From here to the end of this article we assume that the functions \( \rho, a : [r_0, r_1] \to \mathbb{R} \) satisfy the following conditions:
\[ \rho \in C^1[0, r_1], \quad \rho'' \in L^2(r_0, r_1), \quad \rho(r) > 0 \quad \text{on } [r_0, r_1], \quad (1.9) \]
\[ a \in C^2[0, r_1], \quad a(r) > 0 \quad (r \in [r_0, r_1]). \quad (1.10) \]

The following lemma shows that under these assumptions we can reduce the problem (1.1)–(1.5) to a one-dimensional nonstandard boundary-value problem for an ordinary differential equation of second order. Here “nonstandard” refers to the fact that the spectral parameter appears in the boundary conditions.

**Lemma 1.4.** The transmission eigenvalues \( \lambda \) of (1.1)–(1.5) for which there exist spherically-symmetric and continuous wave functions \((w, \nu)\) in \((\overline{D}_1 \setminus D_0) \times D_1\) coincide with the eigenvalues of the nonstandard boundary-value problem
\[ \psi''(x) + (\lambda n(x) + b(x))\psi(x) = 0 \quad (x \in [0, r_1 - r_0]), \quad (1.11) \]
\[ \psi(0) = 0, \quad (1.12) \]
\[ a(r) \frac{\sin \sqrt{\lambda} r_1}{\sqrt{\lambda}} \psi'(r_1 - r_0) - \left[ a \frac{\sin \sqrt{\lambda} r_1}{\sqrt{\lambda} r_1} + \cos \sqrt{\lambda} r_1 \right] \psi(r_1 - r_0) = 0, \quad (1.13) \]
where
\[ n(x) = \frac{\rho(x + r_0)}{a(x + r_0)} \quad (x \in [0, r_1 - r_0]), \quad (1.14) \]
\[ b(x) = \frac{a''(x + r_0)}{2a(x + r_0)} - \frac{a'(x + r_0)}{(x + r_0)a(x + r_0)} - \frac{3}{4} \left( \frac{a'(x + r_0)}{a(x + r_0)} \right)^2 \quad (x \in [0, r_1 - r_0]), \quad (1.15) \]
\[ a = a(r_1) + \frac{r_1}{2} a'(r_1) - 1. \]
Proof. Let us introduce the spherical coordinates \((r, \varphi, \theta)\) by the formulae
\[
x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta.
\]
By the well-known expressions of the gradient and the Laplacian in spherical coordinates we have
\[
\nabla w = w'(r)\hat{r},
\]
\[
\nabla \cdot (a(r)\nabla w) = a(r)\nabla \cdot \nabla w + \nabla a(r) \cdot \nabla w = a(r)\Delta w + a'(r)\hat{r} \cdot w'(r)\hat{r} = a(r)\left(\frac{w''(r)}{r} \right) + a'(r)w'(r),
\]
where \((\hat{r}, \hat{\varphi}, \hat{\theta})\) is the spherical orthonormal base. Therefore, on \(\partial D_1\) we have \(a \cdot \nabla w = a(r_1)w'(r_1).\) Hence the problem (1.1)–(1.5) can be rewritten in the form

\[
\begin{aligned}
\left\{
\begin{align*}
a(r)\left(\frac{w''(r)}{r} \right) + a'(r)w'(r) + \lambda \rho(r)w(r) &= 0 \quad (r_0 < r < r_1), \\
\frac{w''(r)}{r} + \frac{2w'(r)}{r} + \lambda v(r) &= 0 \quad (0 < r < r_1), \\
w(r_1) &= v'(r_1), \quad a(r_1)w'(r_1) = v'(r_1), \\
w(r_0) &= 0.
\end{align*}
\right.
\end{aligned}
\]

Our next step will be to get rid of the terms with the first derivatives of \(w, v.\) We make changes of variables \(w, v\) by the formulae
\[
w(r) = \frac{\Phi(r)}{r\sqrt{a(r)}}, \quad v(r) = \frac{\Phi_0(r)}{r},
\]
where \(\Phi, \Phi_0\) are the new unknown functions. After some easy calculations we get
\[
\Phi''(r) + \left(\frac{\rho(r)}{a(r)} - \frac{a'(r)}{ra(r)} - \frac{3}{4} \left(\frac{a'(r)}{a(r)}\right)^2 + \frac{a''(r)}{2a(r)}\right)\Phi(r) = 0 \quad (r_0 < r < r_1),
\]
(1.16)
\[
\Phi''_0(r) + \lambda \Phi_0(r) = 0 \quad (0 < r < r_1),
\]
(1.17)
\[
\Phi(r_1) = \sqrt{a(r_1)}\Phi_0(r_1),
\]
(1.18)
\[
\frac{\sqrt{a(r_1)}}{r_1}\Phi'(r_1) = \Phi_0(r_1) + \left(\frac{a(r_1)}{r_1^2a(r_1)} + \frac{a'(r_1)}{2r_1}\right) = \Phi_0(r_1) + \frac{\Phi_0'(r_1)r_1 - \Phi_0(r_1)}{r_1^2},
\]
(1.19)
\[
\Phi_0(0) = 0, \quad \Phi_0(r_0) = 0.
\]
(1.20)

In view of (1.18) the condition (1.19) is equivalent to
\[
2r_1\Phi'(r_1)\sqrt{a(r_1)} - \Phi_0(r_1)(2a(r_1) + r_1a'(r_1)) = 2\Phi_0'(r_1)r_1 - 2\Phi_0(r_1).
\]
(1.22)
It follows from (1.17), (1.20) that
\[
\Phi_0(r) = C\frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}}
\]
(1.23)
(for \(\sqrt{\lambda}\) we take its principal value). Hence, (1.18) is equivalent to
\[
C = \frac{\Phi(r_1)\sqrt{\lambda}}{\sin(\sqrt{\lambda}r_1)\sqrt{a(r_1)}}.
\]
(1.24)
Using (1.23), (1.24) we can transform (1.22) into the form
\[
a(r_1)\sin(\sqrt{\lambda}r_1)\Phi'(r_1) - \left(\frac{a(r_1)}{r_1^2a(r_1)} + \frac{a'(r_1)}{2r_1}\right)\sin(\sqrt{\lambda}r_1) + \cos(\sqrt{\lambda}r_1)\Phi(r_1) = 0.
\]
(1.25)
So (1.16)–(1.21) is equivalent to (1.16), (1.21), (1.25). Denote \(\psi(x) = \Phi(x + r_0)\ (x \in [0, r_1 - r_0]).\) Then (1.16), (1.21), (1.25) can be rewritten in the form of (1.11)–(1.13). This finishes the proof of the lemma. □
Theorem 1.5. Under the assumptions (1.9), (1.10) and the additional condition
\[ \beta = \int_{r_2}^{r_1} \sqrt{\frac{p(x)}{a(x)}} \, dx \neq r_1, \]  
(1.26)
the spherically-symmetric interior transmission eigenvalue problem (1.1)–(1.5) has an infinite set of transmission eigenvalues.

Proof. Together with (1.11)–(1.13) we will consider the Cauchy problem
\[ \varphi'' + (\lambda n + b)\varphi = 0 \quad (x \in (0, r_1 - r_0)), \]  
(1.27)
\[ \varphi(0) = 0, \quad \varphi'(0) = 1. \]  
(1.28)
From well-known theorems for the Cauchy problem it follows that for any \( \lambda \in \mathbb{C} \) the problem (1.27), (1.28) has a unique solution \( \varphi(x, \lambda) \) and, moreover, \( \varphi(x, \lambda) \) and \( \varphi'(x, \lambda) \) are entire in \( \lambda \). Since \( n \) and \( b \) are real-valued functions, so \( \varphi(x, \lambda) = \varphi(x, \ell) \) for \( x \in (0, r_1 - r_0), \lambda \in \mathbb{C} \). Define
\[ f(\lambda) = a(r_1) \frac{\sin \sqrt{\lambda} r_1}{\sqrt{\lambda}} \varphi(r_1 - r_0, \lambda) - \left[ \frac{\sin \sqrt{\lambda} r_1}{\sqrt{\lambda} r_1} + \cos \sqrt{\lambda} r_1 \right] \varphi(r_1 - r_0, \lambda) \quad (\lambda \in \mathbb{C}). \]  
(1.29)
Obviously, \( f \) is an entire function and the eigenvalues of the problem (1.11)–(1.13) coincide with zeros of \( f \) (for different types of boundary conditions this fact has been noticed in [1, 11, 14] etc.). For \( \lambda \in \mathbb{R} \) obviously \( f(\lambda) \in \mathbb{R} \).

To complete the proof we will show that for any real \( \lambda_0 \) there are infinitely many values \( \lambda > \lambda_0 \) for which \( f(\lambda) > 0 \) and infinitely many values \( \lambda < \lambda_0 \) for which \( f(\lambda) < 0 \). This will imply that \( f \) has infinitely many zeros; hence there exists an infinite set of transmission eigenvalues.

Let us transform the problem (1.27), (1.28) by the well-known Liouville transform. We introduce the new independent variable \( y \) and the new unknown function \( \eta \) as follows (see [7, p. 79]):
\[ y = \int_{0}^{x} \sqrt{n(s)} \, ds, \quad \eta(y) = \varphi(x)(n(x))^{-\frac{1}{2}}. \]  
(1.30)
Then (1.27), (1.28) is replaced with the equivalent problem
\[ \eta'' + (\lambda - p(y))\eta = 0, \]  
(1.31)
\[ \eta(0) = 0, \quad \eta'(0) = (n(0))^{-\frac{1}{2}}, \]  
where
\[ p(y) = \frac{n'(x)}{4n(x)^{\frac{3}{2}}} - \frac{5n'(x)^2}{16n(x)^{\frac{5}{2}}} - b(x) \cdot n(x). \]  
(1.32)
It is easy to substitute the Cauchy problem (1.31) with the equivalent integral equation
\[ \eta(y) = \frac{\sin(\sqrt{\lambda} y)}{\sqrt{\lambda} n(0)^{-\frac{1}{2}}} + \frac{1}{\sqrt{\lambda}} \int_{0}^{y} \sin(\sqrt{\lambda}(s - y))\eta(s) \, p(s) \, ds. \]  
(1.33)
From this equation, using the method of successive approximations, we can get asymptotic formulae for \( \eta \) and \( \eta' \) which, after a change of variables (1.30), obtain the following form (see [4, p. 282]):
\[ \varphi(x, \lambda) = \frac{1}{\sqrt{\lambda} (n(0)n(x))^{\frac{1}{2}}} \sin \left( \sqrt{\lambda} \int_{0}^{x} \sqrt{n(s)} \, ds \right) + O \left( \frac{1}{\lambda} \right) \quad (\lambda \to \infty), \]  
(1.34)
\[ \varphi'(x, \lambda) = \left( \frac{n(x)}{n(0)} \right)^{\frac{1}{2}} \cos \left( \sqrt{\lambda} \int_{0}^{x} \sqrt{n(s)} \, ds \right) + O \left( \frac{1}{\sqrt{\lambda}} \right) \quad (\lambda \to \infty), \]  
(1.35)
where the estimates of the remainders are uniform in \( x \) as \( \lambda \to \infty \). From these asymptotic formulae and (1.29) we get
\[ f(\lambda) = \frac{1}{\sqrt{\lambda} (n(0)n(r_1 - r_0))^{\frac{1}{2}}} \left( \Delta(\lambda) + O \left( \frac{1}{\sqrt{\lambda}} \right) \right) \quad (\lambda \to \infty), \]  
(1.36)
From (1.15) we have

\[
\Delta(\lambda) = a(r_1) \sqrt{n(r_1 - r_0)} \sin(\sqrt{\lambda} r_1) \cos \left( \int_0^{r_1-r_0} \sqrt{n(s)} \, ds \right) - \cos(\sqrt{\lambda} r_1) \sin \left( \int_0^{r_1-r_0} \sqrt{n(s)} \, ds \right).
\]

From (1.36) it is clear that for sufficiently larger \(\lambda\) the sign of \(f(\lambda)\) coincides with the sign of \(\Delta(\lambda)\). With

\[
y = a(r_1) \sqrt{n(r_1 - r_0)} \quad \text{and} \quad \beta = \int_0^{r_1-r_0} \sqrt{n(s)} \, ds
\]

from (1.26) we obtain

\[
\Delta(\lambda^2) = y \sin(r_1 \lambda) \cos(\beta \lambda) - \cos(r_1 \lambda) \sin(\beta \lambda).
\]  

(1.37)

Let us distinguish between two cases.

\textbf{Case 1:} \(\frac{\beta}{r_1}\) is a rational number. Then \(\frac{\beta}{r_1} = \frac{p}{q}\) for natural numbers \(p_0\) and \(q_0\). From (1.37) we have

\[
\Delta(q_0^2\lambda^2) = y \sin(q_0 r_1 \lambda) \cos(p_0 r_1 \lambda) - \cos(q_0 r_1 \lambda) \sin(p_0 r_1 \lambda)
\]

\[
= y - \frac{1}{2} \sin((q_0 + p_0) r_1 \lambda) + \frac{1}{2} \sin((q_0 - p_0) r_1 \lambda).
\]

In view of (1.26) we have \(p_0 \neq q_0\) and hence \(\Delta(q_0^2\lambda^2)\) is not identically equal to 0. Taking into the consideration that \(\lambda \mapsto \Delta(q_0^2\lambda^2)\) is a \(\frac{2\pi}{p_0}\)-periodic odd function, we conclude that for any real \(\lambda_0\) there are infinitely many values \(\lambda > \lambda_0\) for which \(f(\lambda) > 0\) and infinitely many values \(\lambda > \lambda_0\) for which \(f(\lambda) < 0\).

\textbf{Case 2:} \(\frac{\beta}{r_1}\) is an irrational number. From here to the end of the proof we use the notation \(|x|\) for the fractional part of the real number \(x\). From (1.37) for any natural \(k\) we have

\[
\Delta\left(\frac{2\pi k}{r_1}\right)^2 = -\sin\left(2\pi \frac{k\beta}{r_1^2}\right) = -\sin\left(2\pi \left\{ \frac{k\beta}{r_1} \right\} \right).
\]

From a well-known theorem from the analytic number theory, namely that for irrational \(\xi\) the set \([k\xi : k \in \mathbb{Z}]\) is uniformly distributed modulo 1, we conclude that the set \([\{k\xi\} : k \in \mathbb{Z}]\) of fractional parts is dense in the interval \([0, 1]\) (see [10, pp. 6–8]). This fact shows that for any real \(\lambda_0\) there are infinitely many values \(\lambda > \lambda_0\) for which \(F(\lambda) > 0\) and infinitely many values \(\lambda > \lambda_0\) for which \(f(\lambda) < 0\). This ends the proof of this theorem. \(\square\)

\section{2 The inverse problem}

Now let us consider the special case of the spherically-symmetric interior transmission eigenvalue problem (1.1)–(1.5) with

\[
a(r_1) = 1, \quad a'(r_1) = 0, \quad (2.1)
\]

\[
\rho(r_1) = 1, \quad \rho'(r_1) = 0, \quad (2.2)
\]

\[
\beta < r_1, \quad \beta \neq r_1 - r_0, \quad (2.3)
\]

where \(\beta\) is defined via (1.26).

\textbf{Our inverse problem is the determination of} \(\rho(x)\) for \(x \in [r_0, r_1]\) \textbf{from the knowledge of the transmission eigenvalues of} (1.1)–(1.5) \textbf{which correspond to the radial symmetric eigenpairs including their multiplicities.}

To be more precise, for given \(a(\cdot)\), satisfying (1.10), (2.1), and given discrete set of zeros of \(f(\lambda)\) with their multiplicities we want to recover \(\rho\) in the class of functions satisfying conditions (1.9), (2.2), (2.3).

We consider only the uniqueness aspect of the inverse problem.

\textbf{Theorem 2.1.} \textit{The inverse problem has at most one solution.}

\textbf{Proof.} From (1.15) we have \(\alpha = 0\). Let us apply the Liouville transform

\[
y = y(x) = \int_0^x \sqrt{n(s)} \, ds, \quad V(y) = \psi(x)(rn(x))^\frac{1}{2} \quad (2.4)
\]
to (1.11)–(1.13) (compare to (1.30)). In view of (1.15), (2.1), (2.2) we have
\[ n(r_1 - r_0) = 1, \quad n'(r_1 - r_0) = 0, \]
and hence
\[ V(\beta) = \psi(r_1 - r_0), \quad V'(\beta) = \psi'(r_1 - r_0). \]

Equations (2.6) show that the Liouville transform does not change the boundary condition at the right endpoint (which is obviously true for the left endpoint 0 as well). Therefore, we get
\[ V''(y) + (\lambda - p(y))V(y) = 0, \tag{2.7} \]
\[ V(0) = 0, \tag{2.8} \]
\[ \frac{\sin(\sqrt{\lambda}r_1)}{\sqrt{\lambda}} V'(\beta) - \cos(\sqrt{\lambda}r_1) V(\beta) = 0, \tag{2.9} \]
where \( p \) is defined via (1.32). This problem is investigated in [12]. In particular, under the condition (2.3) it is proved in [12] that for positive eigenvalues of (2.7)–(2.9) the following asymptotic relation holds:
\[ \lambda_j = \frac{j^2\pi^2}{(\beta + r_0 - r_1)^2} + O(1) \quad (j \to \infty). \tag{2.10} \]

First of all let us notice that if for \( \rho_1 \) and \( \rho_2 \) satisfying (1.9), (2.2), (2.3) we have the same set of positive transmission eigenvalues, then from (2.10) we get \( \beta_1 = \beta_2 \), which we denote by \( \beta \). Further, let us denote by \( \varphi_j(x, \lambda) \) and \( \varphi_j(x, \lambda) \) the solutions of (1.27), (1.28) corresponding to \( \rho_1 \) and \( \rho_2 \), respectively. From the integral equation (1.33) using the same technics as in (1.34), (1.35) we obtain that there exists a constant \( A > 0 \) such that
\[ \left| \varphi_j(x, \lambda) - \frac{1}{\sqrt{\lambda}(n_j(0)n_j(x))^{1/2}} \sin(\sqrt{\lambda}y(x)) \right| \leq \frac{A}{|\lambda|} \exp(|\text{Im} \sqrt{\lambda}|y(x)), \]
\[ \left| \varphi_j'(x, \lambda) - \left( \frac{n_j(x)}{n_j(0)} \right)^{1/2} \cos(\sqrt{\lambda}y(x)) \right| \leq A \exp(|\text{Im} \sqrt{\lambda}|y(x)), \]
where \( y(x) \) is defined via (2.4). Moreover, for any positive \( \varepsilon \) and for any fixed \( x \in [0, r_1 - r_0) \) as \( |\lambda| \to \infty \) in the sector
\[ \mathcal{C}_\varepsilon = \{ \lambda \in \mathbb{C} : \varepsilon \leq \arg(\lambda) \leq 2\pi - \varepsilon \} \]
the following asymptotic relations hold:
\[ \varphi_j(x, \lambda) = \frac{1}{\sqrt{\lambda}(n_j(0)n_j(x))^{1/2}} \sin(\sqrt{\lambda}y(x)) \left[ 1 + O\left( \frac{1}{\sqrt{\lambda}} \right) \right], \]
\[ \varphi_j'(x, \lambda) = \left( \frac{n_j(x)}{n_j(0)} \right)^{1/2} \cos(\sqrt{\lambda}y(x)) \left[ 1 + O\left( \frac{1}{\sqrt{\lambda}} \right) \right], \]
(see [1, 13]). From these relations we obtain that
\[ \left\{ \begin{array}{ll}
\varphi_j(r_1 - r_0, \lambda) = \exp(|\text{Im} \sqrt{\lambda}|\beta)\alpha\left( \frac{1}{\sqrt{\lambda}} \right) & (|\lambda| \to \infty \text{ in } \mathbb{C}), \\
\varphi_j'(r_1 - r_0, \lambda) = \exp(|\text{Im} \sqrt{\lambda}|\beta)\alpha(1) & (|\lambda| \to \infty \text{ in } \mathbb{C}).
\end{array} \right. \tag{2.11} \]
From (2.3), (2.11) we will also get
\[ \left\{ \begin{array}{ll}
\varphi_j(r_1 - r_0, \lambda) = \exp(|\text{Im} \sqrt{\lambda}|r_1)\beta\alpha\left( \frac{1}{\sqrt{\lambda}} \right) & (|\lambda| \to \infty \text{ in } \mathbb{C}), \\
\varphi_j'(r_1 - r_0, \lambda) = \exp(|\text{Im} \sqrt{\lambda}|r_1)\beta\alpha(1) & (|\lambda| \to \infty \text{ in } \mathbb{C}).
\end{array} \right. \tag{2.12} \]
In [14] it is proved that \( \varphi_j(x, \cdot), \varphi_j'(x, \cdot) \) \( (j = 1, 2) \) are functions of order \( \frac{1}{2} \).

Assuming that \( f_j(\lambda) \) and \( f_2(\lambda) \) have the same zeros with the same multiplicities we want to prove that \( \rho_1 \equiv \rho_2 \).
Let $\{\lambda_n\}$ be the common sequence of nonzero zeros of $f_j$, $j = 1, 2$, where any zero is repeated as many times according to its multiplicity. If $f_j(0) = f_2(0) = 0$, that is, 0 is also a zero, then we will denote its multiplicity by $m$, otherwise we set $m = 0$. We order the zeros such that $|\lambda_n| \leq |\lambda_{m+1}|$ for all $n$ and define

$$f_0(\lambda) = \lambda^m \sum_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) (\lambda \in \mathbb{C}).$$ (2.13)

Then Hadamard’s factorization theorem states that the infinite product (2.13) converges and there exist constants $\delta_1$ and $\delta_2$ such that

$$f_j(\lambda) = \delta_j f_0(\lambda) \quad (\lambda \in \mathbb{C}, j = 1, 2)$$

(see [6, p. 289]). Therefore we get

$$\frac{1}{\delta_1} f_1 = \frac{1}{\delta_2} f_2 \quad \text{in } \mathbb{C}$$

which, in view of (1.29), can be rewritten as

$$\cos \sqrt{\lambda} \left( \frac{1}{\delta_1} \varphi_1(r_1 - r_0, \lambda) - \frac{1}{\delta_2} \varphi_2(r_1 - r_0, \lambda) \right) = \sin \sqrt{\lambda} \left( \frac{1}{\delta_1} \varphi_1'(r_1 - r_0, \lambda) - \frac{1}{\delta_2} \varphi_2'(r_1 - r_0, \lambda) \right) \quad (\lambda \in \mathbb{C}).$$

Taking $\lambda = \frac{\pi^2}{r_1^2}$ or $\lambda = \frac{(2n-1)^2 \pi^2}{4r_1^2}$ we get

$$\frac{1}{\delta_1} \varphi_1(r_1 - r_0, \frac{n^2 \pi^2}{r_1^2}) - \frac{1}{\delta_2} \varphi_2(r_1 - r_0, \frac{n^2 \pi^2}{r_1^2}) = 0 \quad (n = 1, 2, \ldots),$$

$$\frac{1}{\delta_1} \varphi_1'(r_1 - r_0, \frac{(2n-1)^2 \pi^2}{4r_1^2}) - \frac{1}{\delta_2} \varphi_2'(r_1 - r_0, \frac{(2n-1)^2 \pi^2}{4r_1^2}) = 0 \quad (n = 1, 2, \ldots).$$

From the last two equalities combined with (2.11), (2.12) and analyticity of $\varphi_j(x, \cdot), \varphi_j'(x, \cdot) \quad (j = 1, 2)$ similarly as in [1] it is not difficult to conclude that

$$\frac{1}{\delta_1} \varphi_1^{(j)}(r_1 - r_0, \lambda) = \frac{1}{\delta_2} \varphi_2^{(j)}(r_1 - r_0, \lambda) \quad (\lambda \in \mathbb{C}, j = 0, 1).$$ (2.14)

Equalities (2.14) imply that for any real $\tau$ the eigenvalues of the problem

$$\begin{align*}
\frac{d^2}{d\tau^2} + (\lambda t_1 + b) u &= 0, \\
u(0) &= 0,
\end{align*}$$ (2.15)

coincide with the ones of the problem

$$\begin{align*}
\frac{d^2}{d\tau^2} + (\lambda t_2 + b) u &= 0, \\
u(0) &= 0,
\end{align*}$$ (2.16)

since the eigenvalues of (2.15) are the values of $\lambda$ for which

$$\varphi_1(r_1 - r_0, \lambda) \cos \tau + \varphi_1'(r_1 - r_0, \lambda) \sin \tau = 0$$

and the eigenvalues of (2.16) are the values of $\lambda$ for which

$$\varphi_2(r_1 - r_0, \lambda) \cos \tau + \varphi_2'(r_1 - r_0, \lambda) \sin \tau = 0$$

(see [11–13]). Further, applying the Liouville transform (2.4) to (2.15) and (2.16) we get that the eigenvalues of the problem

$$\begin{align*}
\frac{d^2}{dy^2} + (\lambda - p_1(y)) V(y) &= 0, \\
V(0) &= 0,
\end{align*}$$ (2.17)

$$V(\beta) \cos \tau + V'(\beta) \sin \tau = 0$$
coincide with the ones of the problem

\[
\begin{align*}
V''(y) + (\lambda - p_2(y))V(y) &= 0, \\
V(0) &= 0, \\
V(\beta) \cos \tau + V'(\beta) \sin \tau &= 0,
\end{align*}
\]

(2.18)

where

\[
p_j(y) = \frac{n_j''}{4n_j^2} - \frac{5n_j^2}{16n_j^2} - \frac{b}{n_j} \quad (j = 1, 2).
\]

(2.19)

Now we can apply the fundamental uniqueness theorem of the inverse theory for regular Sturm–Liouville operators, according to which the knowledge of two discrete sets of spectra uniquely determines the coefficient of the Sturm–Liouville operator (see [11–13] and [9, p. 151]). We get \( p_1 = p_2 \) almost everywhere in \((0, \beta)\). Let us denote the common value of \( p_1, p_2 \) by \( p \). From (2.5), (2.19) it follows that \( n_1, n_2 \) are solutions to the Cauchy problem

\[
\begin{align*}
n''(r_1 - r_0) &= 1, \\
n'(r_1 - r_0) &= 0, \\
\end{align*}
\]

therefore, \( n_1 = n_2 \), and the theorem is proved.

**Remark 2.2.** All obtained above results remain true if in (1.1) we substitute \( \lambda \rho(x) \) with \( \lambda - \rho(x) \), i.e. if we consider the Schrödinger equation. In that case some arguments can be simplified, since we do not have to use the Liouville transform and hence we do not need the conditions (2.1) in Schrödinger case.

**Remark 2.3.** In the case \( b \equiv 0 \) (in particular, for constant \( a \)) for \( \lambda = 0 \) from (1.27), (1.28) we have \( \varphi(x, 0) \equiv x \), so \( \varphi'(x, 0) = 1 \), and from (1.29), (2.1) we get \( D(0) = r_0 \neq 0 \), so \( \lambda = 0 \) is not an eigenvalue. Further, note that in the case of constant coefficients \( a, \rho \) (i.e. \( a \equiv \rho \equiv 1 \) in view of (2.1)) we will have \( n \equiv 1, b \equiv 0 \) and hence from (1.27), (1.28) we get \( \varphi(x, \lambda) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \). Therefore, from (1.29) we get \( f(\lambda) \equiv \frac{\sin \sqrt{\lambda} r_0}{\sqrt{\lambda}} \). So, in this case the zeros of \( f(\lambda) \) are simple.

**References**


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