A Note on Sylvester’s Proof of Discreteness of Interior Transmission Eigenvalues

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Title
Une remarque sur la preuve de la distribution discrète des valeurs propres intérieures de transmission de Sylvester

Resume
Dans [10] il a été démontré que l’ensemble des valeurs propres intérieures de transmission constitue un ensemble discret si le contraste ne change pas de signe dans un voisinage du bord. Nous donnons une preuve plus élémentaire de ce fait en utilisant les conditions classiques “inf sup” de Babuška–Brezzi.

Abstract
In [10] it has been shown that the set of interior transmission eigenvalues forms a discrete set if the contrast does not change its sign in a neighborhood of the boundary. In this short note we give a more elementary proof of this fact using the classical inf-sup conditions of Babuška–Brezzi.

1 Introduction

Transmission eigenvalue problems are non-selfadjoint problems which occur in the study of the scattering of time-harmonic waves by inhomogeneous media. The scalar case in
acoustics leads to the problem to determine $k > 0$ and corresponding nontrivial pairs $(u, w)$ such that
\begin{align*}
\Delta w + k^2 w &= 0 \text{ in } D, \quad \Delta u + k^2 (1 + q) u = 0 \text{ in } D, \\
u &= w \text{ on } \partial D, \quad \partial u / \partial \nu = \partial w / \partial \nu \text{ on } \partial D.
\end{align*}
(1.1) (1.2)

As discussed in, e.g., [7] this problem is neither self-adjoint nor elliptic. Therefore, standard results from functional analysis don’t apply. The first question, answered in many papers starting with [3], concerns the discreteness of the spectrum. The assumption that the contrast $q$ does not change its sign in the domain $D$ has been weakened in [10] to the assumption that it does not change its sign on some neighborhood of the boundary $\partial D$.

For the, in some sense simpler (because elliptic) anisotropic case, this has been assumed in, e.g., [1, 6]. For an overview on transmission eigenvalue problems we refer to [2, 5, 9] (see also [7]).

In this note we want to show that Sylvester’s result [10] can also be obtained by the use of the classical inf-sup conditions of Babuška–Brezzi (see [8]) which are closely related to the $T$-coercivity approach of, e.g., [1].

\section{Discreteness of the Spectrum}

As it is well known the eigenvalue problem (1.1), (1.2) is degenerated in the sense that we look for $u, w \in L^2(D)$ such that $u - w \in H^2_0(D) = \{ v \in H^2(D) : v = \partial v / \partial \nu = 0 \text{ on } \partial D \}$. We set $\lambda = -k^2$ and $v = (u - w) / \lambda$. Then the problem is to determine $\lambda \in \mathbb{C}$ and a nontrivial pair $(v, w) \in H^2_0(D) \times L^2(D)$ such that
\begin{align*}
\Delta w - \lambda w &= 0 \quad \text{and} \quad \Delta v - \lambda (1 + q) v = q w \quad \text{in } D
\end{align*}
(2.3)
in the following sense:
\begin{align*}
\int_D [\Delta \bar{v} - \lambda \bar{v}] w \, dx &= 0, \\
\int_D [\Delta \bar{v} - \lambda (1 + q) v - q w] \phi \, dx &= 0
\end{align*}
for all $\psi \in H^2_0(D)$ and $\phi \in L^2(D)$.

\textbf{Definition 2.1} \textit{$\lambda \in \mathbb{C}$ is called interior transmission eigenvalue if there exists a nontrivial pair $(v, w) \in X = H^2_0(D) \times L^2(D)$ such that (2.3) holds in the variational sense.}

We equip $X$ with the norm $\|(v, w)\|_X = \|v\|_{H^2(D)} + \|w\|_{L^2(D)}$ and the corresponding inner product $\langle \cdot, \cdot \rangle_X$. 2
For any \( \lambda \in \mathbb{C} \) we define the sesquilinear form \( a_\lambda : X \times X \to \mathbb{C} \) by
\[
a_\lambda(v, w; \psi, \phi) = \int_D (\Delta \overline{v} - \lambda \overline{\psi}) w \, dx + \int_D (\Delta v - \lambda (1 + q)v) \overline{\phi} - qw \, \overline{\phi} \, dx
\]
for \( (v, w) \in X \) and \( (\psi, \phi) \in X \).

Then \( \lambda \) is an eigenvalue if there exists a nontrivial pair \( (v, w) \in X \) with \( a_\lambda(v, w; \psi, \phi) = 0 \) for all \( (\psi, \phi) \in X \).

We define also the following auxiliary form \( \hat{a}_\lambda \) by
\[
\hat{a}_\lambda(v, w; \psi, \phi) = \int_D (\Delta \overline{\psi} - \lambda \overline{\psi}) w \, dx + \int_D (\Delta v - \lambda v) \overline{\phi} - qw \, \overline{\phi} \, dx
\]
for \( (v, w), (\psi, \phi) \in X \). The representation theorem of Riesz yields the existence of bounded operators \( A_\lambda, \hat{A}_\lambda : X \to X \) such that
\[
a_\lambda(v, w; \psi, \phi) = \langle A_\lambda(v, w); (\psi, \phi) \rangle_X \quad \text{for all } (v, w), (\psi, \phi) \in X, \tag{2.4}
\]
and, analogously, the operator \( \hat{A}_\lambda \) is defined. We note that \( \lambda \) is an eigenvalue if, and only if, \( A_\lambda \) fails to be injective.

We make the following assumption:

**Assumption 2.2** There exists \( q_0 > 0 \) and some neighborhood\(^1\) \( R \) of \( \partial D \) such that \( q \geq q_0 \) on \( R \) or \( q \leq -q_0 \) on \( R \).

We will need the following lemma from the theory of the Helmholtz equation.

**Lemma 2.3** Let \( q \in L^\infty(D) \) satisfy Assumption 2.2. Then there exist \( \hat{c} > 0 \) and \( d > 0 \) such that for all \( \lambda > 0 \) the following estimate holds:
\[
\int_{D \setminus R} |w|^2 \, dx \leq \hat{c} e^{-2d \sqrt{\lambda}} \int_R |q| |w|^2 \, dx \tag{2.5}
\]
for all solutions \( w \in L^2(D) \) of \( \Delta w - \lambda w = 0 \) in \( D \).

**Proof:** We choose a neighborhood \( R' \) of \( \partial D \) with \( d = \text{dist}(D \setminus R, R') > 0 \) and a function \( \rho \in C^\infty(D) \) with compact support in \( D \) and \( \rho = 1 \) in \( D \setminus R' \). We apply Green’s representation theorem (see, e.g., [4]) to \( \rho w \) in \( D \) where \( w \) satisfies \( \Delta w - \lambda w = 0 \) in \( D \) which

\(^1\)that is, an open subdomain \( R \subset D \) with \( \partial D \subset \overline{R} \)
yields
\[
\rho(x) \cdot w(x) = - \int_D \left[ \Delta(\rho w)(y) - \lambda(\rho w)(y) \right] \frac{\exp(-\sqrt{\lambda}|x-y|)}{4\pi|x-y|} \, dy
\]
\[
= - \int_{R'} \left[ 2 \nabla \rho(y) \cdot \nabla w(y) + w(y) \Delta \rho(y) \right] \frac{\exp(-\sqrt{\lambda}|x-y|)}{4\pi|x-y|} \, dy
\]
\[
= \int_{R'} \left[ 2 \text{div}_y \left( \frac{\exp(-\sqrt{\lambda}|x-y|)}{4\pi|x-y|} \right) \cdot \nabla w(y) + w(y) \Delta \rho(y) \exp(-\sqrt{\lambda}|x-y|) \right] w(y) \, dy.
\]

For \( x \in D \setminus R \) we conclude that
\[
|w(x)| \leq c_1 e^{-d \sqrt{\lambda}} \int_R |w(y)| \, dy
\]
for some \( c_1 > 0 \) which depends only on \( D, R, R' \), and \( \rho \), and thus
\[
|w(x)|^2 \leq c_1^2 e^{-2d \sqrt{\lambda}} |R| \int_R |w(y)|^2 \, dy \leq \frac{c_1^2 |R|}{q_0} e^{-2d \sqrt{\lambda}} \int_R |q(y)||w(y)|^2 \, dy.
\]

Integration with respect to \( x \) over \( D \setminus R \) yields the assertion. \( \square \)

We show the following inf–sup condition.

**Theorem 2.4** There exists \( \lambda_0 > 0 \) and \( c > 0 \) such that for all \( \lambda \geq \lambda_0 \)
\[
\sup_{(\psi,\phi) \neq 0} \left| \hat{a}_\lambda(v,w;\psi,\phi) \right| \geq c \|(v,w)\|_X \quad \text{for all } (v,w) \in X. \tag{2.6}
\]

**Proof:** We fix \( \lambda_0 \) such that
\[
\int_{D \setminus R} |q||w|^2 \, dx \leq \|q\|_\infty \int_{D \setminus R} |w|^2 \, dx \leq \frac{1}{2} \int_R |q||w|^2 \, dx \tag{2.7}
\]
for all solutions of \( \Delta w - \lambda w = 0 \) in \( D \) and all \( \lambda \geq \lambda_0 \). This is possible by the estimate (2.5) of Lemma 2.3. If a constant \( c \) with (2.6) does not exist, there exists a sequence \( (v_j,w_j) \in X \) with \( \|(v_j,w_j)\|_X = 1 \) and
\[
\sup_{(\psi,\phi) \neq 0} \left| \hat{a}_\lambda(v_j,w_j;\psi,\phi) \right| \rightarrow 0, \quad j \rightarrow \infty. \tag{2.8}
\]

There exist weakly convergence subsequences \( w_j \rightharpoonup w \) in \( L^2(D) \) and \( v_j \rightharpoonup v \) in \( H^2(D) \) for some \( (v,w) \in X \). From (2.8) we observe that \( (v,w) \) satisfies \( \Delta w - \lambda w = 0 \) and \( \Delta v - \lambda v = qw \) in \( D \).

In the first part we show again that \( v \) and \( w \) vanish.
From \( \text{Re} \hat{a}_\lambda (v, w; -v, w) = 0 \) we conclude that \( \int_D q \, |w|^2 \, dx = 0 \). The estimate (2.7) yields
\[
\int_R |q| \, |w|^2 \, dx = \left| \int_R q \, |w|^2 \, dx \right| = \left| \int_{D \setminus R} q \, |w|^2 \, dx \right| \\
\leq \int_{D \setminus R} |q| \, |w|^2 \, dx \leq \frac{1}{2} \int_R |q| \, |w|^2 \, dx
\]
and thus \( w = 0 \) on \( R \). Analytic continuation yields \( w = 0 \) in all of \( D \) and thus also \( v = 0 \) by \( 0 = \hat{a}_\lambda (v, w; 0, v) = \int_D (\Delta v - \lambda v) \, \overline{v} \, dx = -\int_D (|\nabla v|^2 + \lambda |v|^2) \, dx \).

In the second part we prove a contradiction.

We choose a neighborhood \( R' \) of \( \partial D \) with closure in \( R \cup \partial D \) and a non-negative function \( \rho_1 \in C^\infty (D) \) with \( \rho_1 = 0 \) in \( D \setminus R \) and \( \rho_1 = 1 \) in \( R' \) and substitute \( \psi = \rho_1 v_j \) and \( \phi = -\rho_1 w_j \) in (2.8). Then, because \( (-\rho_1 w_j, \rho_1 v_j) \) is bounded in \( X \),
\[
\int_R \left[ \Delta (\rho_1 \overline{v_j}) - \lambda \rho_1 \overline{v_j} \right] w_j \, dx - \int_R (\Delta v_j - \lambda v_j) \, \rho_1 \, \overline{w_j} - q \rho_1 |w_j|^2 \, dx
\]
tends to zero, thus
\[
\text{Re} \int_R \left[ 2 \, w_j \nabla \rho_1 \cdot \nabla \overline{v_j} + \nabla \overline{w_j} \Delta \rho_1 + q \, \rho_1 |w_j|^2 \right] \, dx \longrightarrow 0 . \tag{2.9}
\]
Since \( v_j \) converges weakly to zero in \( H^2(D) \) it converges to zero in the norm of \( H^1(D) \). Therefore, the first two terms converge to zero, thus also \( \int_R q \, \rho_1 |w_j|^2 \, dx \to 0 \). Since \( q \) is of one sign on \( R \) and \( |q| \rho_1 \geq q_0 \) on \( R' \) we conclude that \( w_j \) tends to zero in \( L^2(R') \).

Now we choose a third neighborhood \( R'' \) of \( \partial D \) with closure in \( R' \cup \partial D \) and a non-negative function \( \rho_2 \in C^\infty (D) \) with \( \rho_2 = 0 \) in \( R'' \) and \( \rho_2 = 1 \) in \( D \setminus R'' \). We determine \( z_j \in H^2(D) \) with \( \Delta z_j - \lambda z_j = w_j \) in \( D \) and \( z_j = 0 \) on \( \partial D \). We substitute \( \phi = 0 \) and \( \psi = \rho_2 z_j \) in (2.8) which yields (note that \( (\rho_2 z_j) \) is bounded in \( H^2(D) \))
\[
\int_{D \setminus R''} \left[ \Delta (\rho_2 \overline{z_j}) - \lambda \rho_2 \overline{z_j} \right] w_j \, dx \longrightarrow 0 ,
\]
that is,
\[
\int_{D \setminus R''} \rho_2 |w_j|^2 + 2 (\nabla \rho_2 \cdot \nabla \overline{z_j}) w_j + \overline{z_j} \Delta \rho_2 w_j \, dx \longrightarrow 0 .
\]
Since \( w_j \to 0 \) in \( L^2(D) \) we conclude that \( z_j \to 0 \) in \( H^2(D) \) and thus \( z_j \to 0 \) in \( H^1(D) \). Furthermore, we note that \( \rho_2 = 1 \) in \( D \setminus R' \) and thus \( \int_{D \setminus R'} |w_j|^2 \, dx \to 0 \).

Altogether, we have shown that \( w_j \to 0 \) in \( L^2(D) \).

Finally, set \( \psi = 0 \) and \( \phi = (\Delta v_j - \lambda v_j) \) in (2.8) which yields
\[
\frac{1}{\| \Delta v_j - \lambda v_j \|^2_{L^2(D)}} \int_D |\Delta v_j - \lambda v_j|^2 - q \, w_j (\Delta \overline{v_j} - \lambda \overline{v_j}) \, dx \longrightarrow 0
\]

that is,
\[ \|\Delta v_j - \lambda v_j\|_{L^2(D)} - \int_D q w_j \frac{\Delta v_j - \lambda v_j}{\|\Delta v_j - \lambda v_j\|_{L^2(D)}} \, dx \to 0, \]
which implies convergence \( \Delta v_j - \lambda v_j \to 0 \) in \( L^2(D) \). Therefore, \( \Delta v_j \) tends to zero in \( L^2(D) \) which is equivalent to \( v_j \to 0 \) in \( H^2(D) \).

Altogether we have shown \((w_j, v_j) \to 0\) in \( X \) which is impossible since its norm is one.

\[ \square \]

**Corollary 2.5** Let \( \lambda_0 > 0 \) such that the inf-sup condition (2.6) of Theorem 2.4 holds. Then the operator \( \hat{A}_\lambda : X \to X \) is self-adjoint and an isomorphism from \( X \) onto itself.

**Proof:** This follows again from a generalized Lax-Milgram theorem (see, e.g., [8]). Note that the non-degeneracy condition holds as well because \( \hat{a}_\lambda \) is hermitian. \[ \square \]

**Theorem 2.6** For any \( \lambda, \mu \in \mathbb{R} \) the differences \( A_\mu - \hat{A}_\lambda \) and \( A_\mu - A_\lambda \) are compact.

**Proof:** Let \((v_j, w_j) \in X \) converge to zero weakly in \( X \) and let \((\psi, \phi) \in X \) with \( \|\!(\psi, \phi)\!\|_X = 1 \). Note that
\[ (a_\mu - \hat{a}_\lambda)(w_j, v_j; \psi, \phi) = (\lambda - \mu) \int_D \psi w_j \, dx + \int_D \left[ \lambda - \mu(1 + q) \right] v_j \phi \, dx. \]
v_j \to 0 in \( H^2(D) \) implies norm convergence \( v_j \to 0 \) in \( L^2(D) \), and thus
\[ \left| \int_D \left[ \lambda - \mu(1 + q) \right] v_j \phi \, dx \right| \leq \| \lambda - \mu(1 + q) \|_{L^\infty(D)} \| v_j \|_{L^2(D)} \| \phi \|_{L^2(D)} \leq \| \lambda - \mu(1 + q) \|_{L^\infty(D)} \| v_j \|_{L^2(D)}. \]
Furthermore, define \( z_j \in H^1(D) \) with \( \Delta z_j = w_j \) in \( D \) and \( z_j = 0 \) on \( \partial D \). Then \( z_j \to 0 \) in \( H^1(D) \) and thus \( z_j \to 0 \) in \( L^2(D) \). Therefore,
\[ \left| \int_D \psi w_j \, dx \right| = \left| \int_D \psi \Delta z_j \, dx \right| = \left| \int_D \Delta \psi z_j \, dx \right| \leq \| \Delta \psi \|_{L^2(D)} \| z_j \|_{L^2(D)} \leq \| z_j \|_{L^2(D)} \]
and altogether
\[ \sup_{\|\!(\psi, \phi)\!\|_X = 1} \left| (a_\mu - \hat{a}_\lambda)(v_j, w_j; \psi, \phi) \right| \leq c \left[ \| z_j \|_{L^2(D)} + \| v_j \|_{L^2(D)} \right] \to 0. \]
This implies compactness of \( A_\mu - \hat{A}_\lambda \). The proof for \( A_\mu - A_\lambda \) follows the same lines. \[ \square \]

**Theorem 2.7** For sufficiently large \( \lambda > 0 \) the operator \( A_\lambda \) is an isomorphism from \( X \) onto itself.
Proof: It is sufficient to prove injectivity because $\hat{A}_\lambda$ is an isomorphism and $\hat{A}_\lambda - A_\lambda$ is compact.

Assume that there exists a sequence $\lambda_j \to \infty$ and functions $(v_j, w_j) \in X$ with $\| (v_j, w_j) \|_X = 1$ and $A_{\lambda_j}(v_j, w_j) = 0$. Therefore, the functions $w_j \in L^2(D)$ and $v_j \in H^2_0(D)$ satisfy the equations

$$\Delta w_j - \lambda_j w_j = 0 \quad \text{and} \quad \Delta v_j - \lambda_j (1 + q)v_j = qw_j \quad \text{in } D. \quad (2.10)$$

Defining $\rho_j = \|q\|_\infty \hat{c} \exp(-2d\sqrt{\lambda_j})$ and splitting the region of integration into $R$ and $D \setminus R$ yields by Lemma 2.3 that

$$(1 - \rho_j) \int_R |q||w_j|^2 dx \leq \int_D q|w_j|^2 dx \leq (1 + \rho_j) \int_R |q||w_j|^2 dx. \quad (2.11)$$

Multiplication of the second equation of (2.10) by $w_j$, integrating and using Green’s second theorem yields

$$\int_D q_{\lambda_j}v_j + w_j dx = 0. \quad (2.12)$$

Multiplication of the second equation of (2.10) by $v_j$, integrating and using Green’s first theorem yields

$$\int_D [|\nabla v_j|^2 + \lambda_j (1 + q)|v_j|^2] dx = -\int_D q w_j v_j dx = \frac{1}{\lambda_j} \int_D q|w_j|^2 dx. \quad (2.13)$$

Now we distinguish between two cases.

Case 1: $q$ is negative on $R$. Then the right integral in (2.13) is negative as it follows from Lemma 2.3 because

$$-\int_D q|w_j|^2 dx \geq -\int_R q|w_j|^2 dx - \int_{D \setminus R} |q||w_j|^2 dx \geq (1 - \rho_j) \int_R |q||w_j|^2 dx > 0.$$

This contradicts (2.13).

Case 2: $q$ is positive on $R$. From (2.12) we conclude

$$(1 - \rho_j) \int_R q|w_j|^2 dx \leq \int_D q|w_j|^2 dx = -\lambda_j \int_D q w_j v_j dx \leq \lambda_j \int_{D \setminus R} |q||w_j||v_j| dx + \lambda_j \int_R q|w_j||v_j| dx \leq \lambda_j \left[ \int_{D \setminus R} |q||w_j|^2 dx \int_{D \setminus R} |q||v_j|^2 dx \right]^{1/2} + \lambda_j \left[ \int_R q|w_j|^2 dx \int_R q|v_j|^2 dx \right]^{1/2} \leq \lambda_j \sqrt{\int_R q|w_j|^2 dx} \left[ \rho_j \|q\|_\infty^2 + \int_R q|v_j|^2 dx \right]^{1/2}.$$

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where we used that $\int_{D \setminus R} |q| |v_j|^2 \, dx \leq \|q\|_\infty$. Therefore, we conclude that

$$\sqrt{\int_R q |w_j|^2 \, dx} \leq \frac{\lambda_j}{1 - \rho_j} \left( \rho_j \sqrt{\|q\|_\infty} + \sqrt{\int_R q |v_j|^2 \, dx} \right).$$

Now we square and use the estimate $(a+b)^2 \leq (1+1/\rho_j)a^2+(1+\rho_j)b^2 = (1+\rho_j)[a^2/\rho_j+b^2]$ for obvious meaning of $a$ and $b$. We arrive at

$$\int_R q |w_j|^2 \, dx \leq \frac{(1+\rho_j)^2 \lambda_j^2}{(1-\rho_j)^2} \left[ \rho_j \|q\|_\infty + \int_R q |v_j|^2 \, dx \right].$$

We substitute this for the right hand side of (2.13):

$$\lambda_j \int_R (1 + q) |v_j|^2 \, dx \leq \frac{1+\rho_j}{\lambda_j} \int_R q |w_j|^2 \, dx \leq \frac{(1+\rho_j)^2 \lambda_j^2}{(1-\rho_j)^2} \left[ \rho_j \|q\|_\infty + \int_R q |v_j|^2 \, dx \right]$$

and thus

$$\int_R |v_j|^2 \, dx \leq \frac{(1+\rho_j)^2}{(1-\rho_j)^2} \rho_j \|q\|_\infty + \left( \frac{(1+\rho_j)^2}{(1-\rho_j)^2} - 1 \right) \left( \int_R q |v_j|^2 \, dx \right) \leq \|q\|_\infty$$

$$\leq \frac{(1+\rho_j)^2}{(1-\rho_j)^2} \rho_j \|q\|_\infty + 4 \rho_j \|q\|_\infty \leq c_1 \rho_j$$

for some $c_1 > 0$. From (2.14) and the observation that $\lambda_j^2 \rho_j \to 0$ we note that $\int_R q |w_j|^2 \, dx$ tends to zero and thus also $w_j \to 0$ in $L^2(D)$ by Lemma 2.3. Finally, from the (2.13) and the assumption $1 + q \geq q_1 > 0$ we conclude that $q_1 \lambda_j^2 \|v_j\|_{L^2(D)} \leq \int_D |q| |w_j|^2 \, dx \to 0$; that is, $\lambda_j v_j$ tends to zero in $L^2(D)$. Now we use the continuous dependence of the solution of $\Delta v_j = \lambda_j v_j + qw_j$ which yields that $v_j$ tends to zero in $H^2(D)$, a contradiction to $\|(v_j, w_j)\|_X = 1$. 

Therefore, as in the previous section we fix $\lambda_0 > 0$ such that $A_{\lambda_0}$ is an isomorphism and rewrite the equation $A_{\lambda}(v, w) = 0$ in the form

$$(v, w) + A_{\lambda_0}^{-1}(A_{\lambda} - A_{\lambda_0})(v, w) = 0.$$

The observation that $A_{\lambda} - A_{\lambda_0} = (\lambda - \lambda_0)K$ for some compact operator $K$ yields discreteness of the spectrum. We formulate the result as a theorem.

**Theorem 2.8** Let there exist $q_0 > 0$ and some neighborhood $R$ of $\partial D$ such that $q \geq q_0$ on $R$ or $q \leq -q_0$ on $R$. Then the set of transmission eigenvalues is discrete. In $\mathbb{C}$ there is no (finite) accumulation point.
References


