A modification of the factorization method for the classical acoustic inverse scattering problems

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Abstract
It is well-known that sampling type methods for solving inverse scattering problems fail if the wave number is an eigenvalue of a corresponding interior eigenvalue problem. By adding the far field patterns corresponding to an artificial ball lying within the obstacle and imposing an impedance boundary condition on the boundary of this ball we propose a modification of the factorization method which provides the characterization of the unknown obstacle for all wave numbers. Some numerical experiments are presented to demonstrate the feasibility and effectiveness of our method.

Keywords: acoustic scattering, factorization, method, impenetrable obstacle, eigenvalue

(Some figures may appear in colour only in the online journal)

1. Introduction
The factorization method is a sampling method for solving certain kinds of inverse problems where a domain as a geometric parameter has to be determined. It has first been introduced by one of the authors (AK) in [5] for acoustic scattering problems for plane incident waves by sound-soft and sound-hard obstacles. These are exactly the cases which we will treat in this paper. In [6] and, more satisfactorily, in [1, 7, 9] it has been extended to the scattering by an absorbing or non-absorbing inhomogeneous medium. The case of scattering by an impedance

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condition has been treated in [4] and [8]. We also refer to the monograph [8] for a detailed investigation of the factorization method for a large number of inverse problems. We will add some remarks at the end of section 3 concerning the general advantages and drawbacks of the factorization method.

For inverse scattering problems the data of the inverse problems consist of the far field patterns for plane incident waves which define the kernel of far field operator $F$. The injectivity of $F$ (which is equivalent to the denseness of its range) is closely related to a corresponding interior eigenvalue problem. For a rigorous justification of the original factorization method it has to be assumed that the wave number is not an eigenvalue of the Laplacian with respect to the boundary condition of the scattering problem. Therefore, from a theoretical point of view, this is unsatisfactory because the eigenvalues depend on the unknown obstacle and are therefore not known in advance. We want to mention that for penetrable media Lechleiter has proven in [11] that a particular version of the factorization method is indeed independent of interior transmission eigenvalues. It is the aim of this paper to modify the method such that the characterizations of the impenetrable obstacles hold independently whether or not the wave number is an eigenvalue. However, there is a price to pay. For the original method no topological property of the unknown domain $\Omega$ has to be known. For the modification discussed in this paper one has to know the center and radius of a small ball (or any other domain) $B$ in the interior of $\Omega$.

We begin with the formulations of the scattering problems. We restrict ourselves to the three dimensional case, but all of the results hold also in $\mathbb{R}^2$ with possibly different normalizing constants. Let $k = \omega/c > 0$ be the wave number of a time harmonic wave where $\omega > 0$ and $c > 0$ denote the frequency and sound speed, respectively. Let $\Omega \subset \mathbb{R}^3$ be an open and bounded domain with Lipschitz-boundary $\partial \Omega$ such that the exterior $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected. Furthermore, let $u'$ be a plane wave of the form

$$
u'(x) = u'(x, \hat{\nu}) = e^{ikx \cdot \hat{\nu}}, \quad x \in \mathbb{R}^3,$$  \hspace{1cm} (1.1)

where $\hat{\nu} \in S^2$ denotes the direction of the incident wave and $S^2 = \{ x \in \mathbb{R}^3 : |x| = 1 \}$ is the unit sphere in $\mathbb{R}^3$. The direct scattering is to find the scattered field $u' \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\Omega})$ such that the total field $u = u' + u''$—and thus also $u''$—satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}. \hspace{1cm} (1.2)$$

Here, $H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\Omega})$ denotes the ordinary (local) Sobolov space of order one; that is,

$$H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\Omega}) = \{ u : \mathbb{R}^3 \setminus \overline{\Omega} \to \mathbb{C} : u|_{\partial \Omega} \in H^1(\partial \Omega) \}$$

for all open balls $\tilde{B}$. Furthermore, the scattered field $u'$ has to satisfy the Sommerfeld radiation condition

$$\lim_{r \to \infty} r \left( \frac{\partial u'}{\partial r} - iku' \right) = 0 \quad \text{for} \quad r = |x| \to \infty \hspace{1cm} (1.3)$$

uniformly with respect to all directions $x/|x| \in S^2$. Finally, some boundary condition on $\partial \Omega$ has to be satisfied which depends on the intrinsic physical properties of the underlying obstacle. For a sound-soft obstacle the pressure of the total wave $u = u' + u''$ vanishes on the boundary, so a Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial \Omega \hspace{1cm} (1.4)$$

is imposed. Analogously, the scattering from a sound-hard obstacle leads to the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega, \hspace{1cm} (1.5)$$
where $v$ is the unit outward normal to $\partial \Omega$. Also the more general impedance boundary condition of the form
\[ \frac{\partial u}{\partial v} + i\lambda u = 0 \quad \text{on} \quad \partial \Omega \] (1.6)
with some non-negative impedance function $\lambda \in L^\infty(\partial \Omega)$ is widely used. It is well known (see, e.g. [2, 14]) that these scattering problems have unique solutions $u = u(x, \hat{\theta})$ (the dependence on $k$ is suppressed because we keep the wave number fixed).

Every radiating solution of the Helmholtz equation has the following asymptotic behavior at infinity:
\[ u^\epsilon(x) = e^{i|\xi|} \left\{ u^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\} \quad \text{as} \quad |x| \to \infty \] (1.7)
uniformly with respect to all directions $\hat{x} := x/|x| \in S^2$, see, e.g., [3]. The complex valued function $u^\infty = u^\infty(\hat{x}, \hat{\theta})$ is called the far field pattern or scattering amplitude of $u^\epsilon$. We recall that $\hat{x}$ and $\hat{\theta}$ denote the direction of observation and incidence, respectively. With this scattering amplitude we define the far field operator $F : L^2(S^2) \to L^2(S^2)$ by
\[ (Fg)(\hat{x}) = \int_{S^2} u^\infty(\hat{x}, \hat{\theta}) g(\hat{\theta}) \, d\hat{\theta}, \quad \hat{x} \in S^2. \] (1.8)

The inverse scattering problem which we treat in this paper is to determine the domain $\Omega$ from the far field pattern $u^\infty(\hat{x}, \hat{\theta})$ for all $\hat{x}, \hat{\theta} \in S^2$. In other words, given the operator $F$ (and $k$), determine $\Omega$!

In all of the scattering problems mentioned above the far field operator allows a factorization of the form
\[ F = GTG^* \] (1.9)
with different forms of the operators $G$ and $T$, see sections 2 and 3. Here, $G^*$ denotes the adjoint operator of $G$. The operator $G$ is compact and $T$ is an isomorphism. Therefore it is quite different from the spectral decomposition of an operator where $G$ is invertible and $T$ is compact.

The first key observation is that in all of these cases the function
\[ \phi^\approx(\hat{x}) = e^{-ikz\hat{x}}, \quad \hat{x} \in S^2, \] (1.10)
belongs to the range of $G$ if and only if $z \in \Omega$. Therefore, the unknown domain $\Omega$ is characterized by the range of $G$. However, $G$ is not known a priori. The second key ingredient is that one can characterize the range of $G$ by the known operator $F$. This is formulated in the following functional analytic theorem (see theorem 2.15 in [8] for the cases $i = 0$ or $i = \pi$):

**Theorem 1.1.** Let $X \subset U \subset X^*$ be a Gelfand triple with a Hilbert space $U$ and a reflexive Banach space $X$ such that the imbedding is dense. Furthermore, let $Y$ be a second Hilbert space and let $F : Y \to Y$, $G : X \to Y$, and $T : X^* \to X$ be linear bounded operators such that
\[ F = GTG^*. \]
We make the following assumptions:

1. $G$ is compact with dense range.
2. There exists $\sigma \in [-1, 1]$ such that $\Re T := \frac{i}{2}(T + T^*)$ has the form $\Re T = \sigma C + K$ with some compact operator $K$ and some self adjoint and coercive operator $C : X^* \to X$; that is, there exists $c > 0$ with
\[ \langle \phi, C\phi \rangle \geq c\|\phi\|^2 \quad \text{for all} \quad \phi \in X^*.\]
Theorem 2.6, the following factorization has been proven. A factorization for the impedance boundary condition (3) \( \exists T := \frac{1}{2}(T - T^*) \) is non-negative on \( \mathcal{R}(G^*) \subset X^* \), i.e., \( \exists \langle \phi, T\phi \rangle \geq 0 \) for all \( \phi \in \mathcal{R}(G^*) \).

(4) \( \exists T \) is positive on the closure \( \overline{\mathcal{R}(G^*)} \) of \( \mathcal{R}(G^*) \), i.e., \( \exists \langle \phi, T\phi \rangle > 0 \) for all \( \phi \in \mathcal{R}(G^*) \), \( \phi \neq 0 \). Then the self-adjoint operator \( F_z = |\Re F| + \Im F \) is positive, and the ranges of \( G : X \to Y \) and \( F_z^{1/2} : Y \to Y \) coincide.

In the application to the scattering problems it is the assumption (4) which requires to exclude certain eigenvalues. In the scattering problem with Dirichlet or Neumann boundary conditions these are the eigenvalues \( k^2 \) of \(-\Delta \) in \( \Omega \) with respect to Dirichlet or Neumann boundary conditions, respectively.

This paper is organized as follows. In section 2, we recall the factorization method for the impedance boundary condition (1.6) for any positive constant impedance function \( \lambda = \lambda_0 \). This result is needed as an auxiliary tool for section 3. Assumption (4) of theorem 1.1 can be proven because there do not exist real eigenvalues of \(-\Delta \) with respect to the impedance boundary condition. Section 3 is devoted to a modification of the factorization method for impenetrable obstacles. We will treat both the Dirichlet and Neumann boundary conditions because in these cases the factorization method fails if the wave number is an eigenvalue of \(-\Delta \) in \( \Omega \) with respect to the corresponding boundary condition. Some numerical simulations in two dimensions will be presented in section 4 to justify the validity of our method.

2. A factorization for the impedance boundary condition

In this section we briefly recall the factorization method for the scattering problem (1.2), (1.3), and (1.6) of plane incident waves by an obstacle under impedance boundary conditions. We consider only the special case of a constant and positive impedance function; that is, \( \lambda = \lambda_0 \in \mathbb{R} \) for some \( \lambda_0 > 0 \). Also, we denote the scattering domain by \( B \) instead of \( \Omega \) because we will use the results of this section later for an auxiliary ball \( B \). This scattering problem is a special case of the more general boundary value problem for given \( f \in H^{1/2}(\partial B) \):

\[
\Delta v + k^2\varepsilon v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B},
\]

\[
\frac{\partial v}{\partial \nu} + i\lambda_0 v = f \quad \text{on } \partial B,
\]

and \( v \in H_{loc}^{1/2}(\mathbb{R}^3 \setminus \overline{B}) \) satisfies the Sommerfeld radiation condition (1.3). The special case \( f = -\partial u'/\partial \nu - i\lambda_0 u' \) corresponds to the scattering problem (1.2), (1.3), (1.6). It is well known that the boundary value problem (2.1)–(2.2) has a unique solution \( v \in H_{loc}^{1/2}(\mathbb{R}^3 \setminus \overline{B}) \) for all \( f \in H^{-1/2}(\partial B) \) where the solution is understood in the variational sense. Again, we denote the far field patterns of \( v \) and \( u' \) by \( v^\infty = v^\infty(\hat{x}, f) \) and \( u^\infty = u^\infty(\hat{x}, \hat{\theta}) \), respectively. We define the data-to-pattern operator \( G : H^{-1/2}(\partial B) \to L^2(S^2) \) by

\[
Gf = v^\infty.
\]

The corresponding far field operator \( F : L^2(S^2) \to L^2(S^2) \) is again given by (1.8). In [8], theorem 2.6, the following factorization has been proven.

**Theorem 2.1.** Let \( F : L^2(S^2) \to L^2(S^2) \) and \( G : H^{-1/2}(\partial B) \to L^2(S^2) \) be given as above. Then the far field operator can be written in the forms

\[
F = -GT^*G^* = -H^*T^{-1}H
\]
where \( T : H^{1/2}(\partial B) \to H^{-1/2}(\partial B) \) and \( H : L^2(S^2) \to H^{-1/2}(\partial B) \) are defined by

\[
(T\phi)(x) = i\lambda_0\phi(x) + \frac{\partial}{\partial v} \int_{\partial B} \phi(y) \frac{\partial}{\partial v(y)} \Phi(x, y) \, ds(y) + \lambda_0^2 \int_{\partial B} \phi(y) \Phi(x, y) \, ds(y)
+ i\lambda_0 \int_{\partial B} \phi(y) \left[ \frac{\partial}{\partial v(y)} \Phi(x, y) - \frac{\partial}{\partial v(x)} \Phi(x, y) \right] \, ds(y), \quad x \in \partial B,
\]

(2.5)

\[
(Hg)(x) = \left( \frac{\partial}{\partial v} + i\lambda_0 \right) \int_{S^2} e^{ik\hat{g}(\theta)} \, ds(\hat{\theta}), \quad x \in \partial B,
\]

(2.6)

respectively. Here, \( \Phi \) denotes the fundamental solution of the Helmholtz equation in \( \mathbb{R}^3 \); that is,

\[
\Phi(x, y) = \frac{\exp(ik|x-y|)}{4\pi|x-y|}, \quad x \neq y.
\]

In (2.4), the operators \( G^*: L^2(S^2) \to H^{1/2}(\partial B) \) and \( T^*: H^{1/2}(\partial B) \to H^{-1/2}(\partial B) \) and \( H^*: H^{1/2}(\partial B) \to L^2(S^2) \) denote the adjoint operators of \( G, T, \) and \( H, \) respectively. Furthermore, the operator \( T \) is an isomorphism from \( H^{1/2}(\partial B) \) onto \( H^{-1/2}(\partial B) \) and is of the form \( T = T_0 + K \) with self adjoint and coercive operator \( T_0 : H^{1/2}(\partial B) \to H^{-1/2}(\partial B) \) and compact operator \( K \).

No eigenvalues \( k^2 \) with respect to the impedance boundary condition (1.6) exist because of \( \lambda_0 > 0 \). This is the reason why the operator \( T \) is always an isomorphism for all \( k > 0 \).

The following lemma on the operator \( 3T \) can be found as lemma 2.7 in [8]. However, we will prove it in a more direct way.

**Lemma 2.2.** The operator \( 3T := \frac{1}{3}(T - T^*) \) is strictly positive; that is,

\[
\Re(T\phi, \phi) > 0 \quad \text{for all } \phi \in H^{1/2}(\partial B), \quad \phi \neq 0.
\]

(2.7)

**Proof.** Define \( v \) by

\[
v(x) = \int_{\partial B} \phi(y) \left( \frac{\partial}{\partial v(y)} \Phi(x, y) - i\lambda_0 \Phi(x, y) \right) \, ds(y), \quad x \notin \partial B.
\]

Then, by the well known jump relations, we have

\[
\phi = v_+ - v_-, \quad i\lambda_0 \phi = \frac{\partial v_+}{\partial v} - \frac{\partial v_-}{\partial v}, \quad \text{and} \quad T\phi = \frac{\partial v_+}{\partial v} + i\lambda_0 v_+,
\]

(2.8)

where the subscripts + and − denote the traces from the exterior and interior, respectively.

From this, we conclude

\[
(T\phi, \phi) = \int_{\partial B} \left( \frac{\partial v_+}{\partial v} + i\lambda_0 v_+ \right) (\bar{v}_+ - \bar{v}_-) \, ds
= \int_{\partial B} \left[ \frac{\partial v_+}{\partial v} + i\lambda_0 |v_+|^2 - \frac{\partial v_+}{\partial v} \bar{v}_- - i\lambda_0 v_+ \bar{v}_- \right] \, ds
= \int_{\partial B} \left[ \frac{\partial v_+}{\partial v} + i\lambda_0 |v_+|^2 \left( \frac{\partial v_-}{\partial v} + i\lambda_0 v_- - i\lambda_0 v_- \right) \bar{v}_- - i\lambda_0 v_+ \bar{v}_- \right] \, ds
= \int_{\partial B} \left[ \frac{\partial v_+}{\partial v} + i\lambda_0 |v_+|^2 - \frac{\partial v_-}{\partial v} \bar{v}_- - 2i\lambda_0 v_+ \bar{v}_- + i\lambda_0 |v_-|^2 \right] \, ds
= \int_{\partial B} \left[ \frac{\partial v_+}{\partial v} + i\lambda_0 |v_+|^2 - \frac{\partial v_-}{\partial v} \bar{v}_- - 2i\lambda_0 v_+ \bar{v}_- - i\lambda_0 v_+ \bar{v}_- \right] \, ds
\]

(2.9)
and thus taking the imaginary part
\[ \Im(\mathcal{L}_0 \phi, \phi) = \Im \int_{\partial B} \frac{\partial v_+}{\partial v} v_+ - \frac{\partial v_-}{\partial v} v_- \, ds + \lambda_0 \int_{\partial B} |v_+ - v_-|^2 \, ds \]
\[ = \Im \int_{|x| = r} \frac{\partial v}{\partial v} \, ds - \Im \int_{|x| < r} |\nabla v|^2 - k^2|v|^2 \, dx + \lambda_0 \int_{\partial B} |v_+ - v_-|^2 \, ds \]
\[ = \Im \int_{|x| = r} \frac{\partial v}{\partial v} \, ds + \lambda_0 \int_{\partial B} |v_+ - v_-|^2 \, ds \]

(2.10)

where \( r \) is large enough such that \( B \) is contained in a ball with radius \( r \). Letting \( r \) tend to infinity yields
\[ \Im(\mathcal{T}_0 \phi, \phi) = \frac{k}{(4\pi)^2} \int_{S^1} |v^\infty|^2 \, ds + \lambda_0 \int_{\partial B} |v_+ - v_-|^2 \, ds. \]

From this we observe that \( \Im(\mathcal{T}_0 \phi, \phi) \geq 0 \) and vanishes if and only if \( v_+ = v_- \) which is equivalent to \( \phi = 0 \).

3. The modified factorization method

As mentioned in the introduction we will now modify the classical factorization method for the scattering of plane waves by an impenetrable obstacle with Dirichlet or Neumann boundary conditions. We will carry out the case of the Dirichlet boundary condition in more detail.

3.1. The Dirichlet boundary condition

First, we consider the case of Dirichlet boundary conditions. By [8] the far field operator \( F : L^2(S^2) \to L^2(S^2) \) can be factorized as
\[ F = -G_{DS^*} G_*^D \]  
(3.1)

where \( S : H^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega) \) denotes the single layer boundary operator, defined as
\[ (S\phi)(x) = \int_{\partial \Omega} \phi(y) \Phi(x, y) \, ds(y), \quad x \in \partial \Omega, \]  
(3.2)

for smooth densities. Again, \( \Phi \) denotes the fundamental solution of the Helmholtz equation. The operator \( G_D : H^{1/2}(\partial \Omega) \to L^2(S^2) \) maps the Dirichlet boundary data \( f \) onto the far field pattern \( v^\infty \) of the corresponding solution \( v \) of the exterior Dirichlet boundary value problem.

Assume that we know an open ball \( B \) such that \( \overline{B} \subset \Omega \). Let \( F_{\lambda_0} \) be the far field operator with respect to \( B \) and the impedance boundary condition (for any fixed \( \lambda_0 > 0 \)) as in the previous section. We note that \( F_{\lambda_0} \) is known once we have chosen \( B \) and \( \lambda_0 \). We can even express the far field pattern for the scattering of plane waves by the ball \( B \) with impedance boundary conditions with constant impedance \( \lambda_0 \) explicitly in terms of spherical harmonics.

By theorem 2.1 we have a factorization of the form
\[ F_{\lambda_0} = -G_{j_\lambda_0} T_{j_\lambda_0}^* G_{j_\lambda_0} \]  
(3.3)

where \( F_{\lambda_0}, G_{\lambda_0}, \) and \( T_{\lambda_0} \) are the operators \( F, G, \) and \( T \) from (1.8), (2.3), and (2.5), respectively. Adding this to (3.1) yields
\[ F + F_{\lambda_0} = -[G_{DS^*} G_{D}^* + G_{j_\lambda_0} T_{j_\lambda_0}^* G_{j_\lambda_0}^*]. \]  
(3.4)

6
Lemma 3.1. The operator $G_{λ_0} : H^{-1/2}(∂B) → L^2(S^2)$ can be written in the form
\[ G_{λ_0} = G_D R_1 \]
where $R_1 : H^{-1/2}(∂B) → H^{1/2}(∂Ω)$ is defined by $R_1 f = v|_Ω$ where $v$ solves the exterior impedance boundary value problem (2.1)–(2.2). Moreover, $R_1$ is compact.

Proof. Let $v$ be the solution of the exterior impedance boundary value problem (2.1)–(2.2) with right hand side $f$. Then $v^∞ = G_{λ_0} f$ by the definition $G_{λ_0}$. On the other hand, also $v^∞ = G_D v|_Ω$ which proves $G_{λ_0} = G_D R_1$. The operator $R_1$ is certainly compact because the mapping $f → v|_Ω$ is bounded from $H^{-1/2}(∂B)$ into $C^m(∂Ω)$ for any $m ∈ η$ by well known interior regularity results.

Therefore, substituting this into (3.4) yields
\[ F + F_{λ_0} = -G_D [S' + R_1 T_{λ_0}^* R_1^*] G_D = -G_D A_1^* G_D^* \]
where $A_1 := S + R_1 T_{λ_0} R_1^* : H^{-1/2}(∂Ω) → H^{1/2}(∂Ω)$. This provides a factorization of $F + F_{λ_0}$. We show that this factorization satisfies all assumptions of theorem 1.1. First we note that $A_1$ is the sum of a self adjoint coercive operator and a compact operator because $S$ has this property (see lemma 1.14 in [8]) and $R_1 T_{λ_0} R_1^*$ is compact. Properties (3) and (4) of theorem 1.1 are shown in the following lemma.

Lemma 3.2.
\[ \Im(A_1 φ, φ) > 0 \quad \text{for all } φ ∈ H^{-1/2}(∂Ω), \quad φ ≠ 0. \]

Proof. First we note that (see [8], lemma 1.14)
\[ \Im(A_1 φ, φ) ≥ 0 \quad \text{for all } φ ∈ H^{-1/2}(∂Ω). \]
By lemma 2.2 we recall that
\[ \Im(R_1 T_{λ_0} R_1^* φ, φ) = \Im(T_{λ_0} R_1^* φ, R_1^* φ) > 0 \]
provided $R_1^* φ ≠ 0$. This proves
\[ \Im(A_1 φ, φ) ≥ 0 \quad \text{for all } φ ∈ H^{-1/2}(∂Ω). \]
Furthermore, $\Im(A_1 φ, φ) = 0$ implies that $R_1^* φ = 0$. It remains to show that $R_1^*$ is one-to-one. We show that $R_1^*$ has the form $R_1^* φ = w|_B$ where $w ∈ H^1_{loc}(R^3 \setminus B)$ solves [13]
\[ \Delta w + k^2 w = 0 \quad \text{in } R^3 \setminus (B ∪ \partialΩ), \]
\[ \frac{∂ w_+}{∂ v} - \frac{∂ w_-}{∂ v} = φ, \quad w_+ - w_- = 0 \quad \text{on } \partialΩ, \]
\[ \frac{∂ w}{∂ v} + iλ_0 w = 0 \quad \text{on } ∂B, \]
and $w$ satisfies the Sommerfeld radiation condition (1.3). Indeed, we have by using the boundary conditions and Green’s theorem (first in the exterior of $D$, then in the region $D \setminus B$),
\[ ⟨R_1 f, φ⟩ = \int_{∂Ω} v φ dσ = \int_{∂Ω} v \left( \frac{∂ w_+}{∂ v} - \frac{∂ w_-}{∂ v} \right) dσ \]
\[ = \int_{∂Ω} \left[ v \left( \frac{∂ w_+}{∂ v} - \frac{∂ w_-}{∂ v} \right) - (w_+ - w_-) \frac{∂ v}{∂ v} \right] dσ \]
\[ = \int_{∂Ω} \left[ w_+ \frac{∂ v}{∂ v} - v \frac{∂ w_-}{∂ v} \right] dσ = \int_{∂B} \left[ w \frac{∂ v}{∂ v} - v \frac{∂ w}{∂ v} \right] dσ \]
\[ = \int_{∂B} [w(f - iλ_0 v) + iλ_0 v w] dσ = \int_{∂B} f w dσ \]
which proves that $R_1^* φ = w|_B$. 7
To show injectivity of \( R^*_1 \) let \( R^*_1 \phi = 0 \). The boundary condition yields \( \partial w/\partial v = 0 \) on \( \partial B \). Holmgren’s uniqueness theorem implies that \( w \) vanishes in \( \Omega \setminus B \). The second transmission condition of (3.7) yields \( w_\pm = 0 \) on \( \partial \Omega \) which implies that \( w \) vanishes also outside of \( \Omega \) by the uniqueness of the exterior Dirichlet problem. The first transmission condition (3.7) yields \( \phi = 0 \) which ends the proof. ⌜

As mentioned in the introduction the domain \( \Omega \) can be characterized by the range of the operator \( G \). Indeed, the following lemma is well known (see [8], theorem 1.12).

**Lemma 3.3.** For \( z \in \mathbb{R}^3 \) let \( \phi_z \in L^2(S^2) \) be defined by (1.10); that is,
\[
\phi_z(\hat{x}) = e^{-i z \cdot \hat{x}}, \quad \hat{x} \in S^2.
\]
(3.9)
Then \( \phi_z \) belongs to the range of \( G_D \) if, and only if, \( z \in \Omega \).

Application of this lemma combined with theorem 1.1 to the factorization (3.5) yields.

**Theorem 3.4.** Define \( F_z : L^2(S^2) \rightarrow L^2(S^2) \) as
\[
F_z = |\Re(F + F_{x_0})| + 3(F + F_{x_0}).
\]
(3.10)
Then \( F_z \) is self adjoint and positive. Furthermore, \( z \in \Omega \) if, and only if, \( \phi_z \) from (3.9) belongs to the range of \( F_z^{1/2} \).

We can reformulate this result as follows.

**Theorem 3.5.** Define \( F_z : L^2(S^2) \rightarrow L^2(S^2) \) as in (3.10). Furthermore, let \( \{\lambda_n, \varphi_n : n \in \mathbb{N}\} \) be a complete eigensystem of the self adjoint and positive operator \( F_z \). Then \( z \in \Omega \) if, and only if,
\[
\sum_{n \in \mathbb{N}} \frac{|(\phi_z, \varphi_n)_{L^2(S^2)}|^2}{\lambda_n} < \infty.
\]
Therefore, the sign of the function
\[
W(z) = \left[ \sum_{n \in \mathbb{N}} \frac{|(\phi_z, \varphi_n)_{L^2(S^2)}|^2}{\lambda_n} \right]^{-1}, \quad z \in \mathbb{R}^3,
\]
is just the characteristic function of \( \Omega \).

### 3.2. The Neumann boundary condition

Now, we turn to the case of the Neumann boundary condition. We only sketch the arguments because they are very similar to the case of Dirichlet boundary conditions. By [8] (theorem 1.16) \( F \) can be factorized as
\[
F = -G_N N^* G_N^*
\]
(3.11)
where \( N : H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega) \) is given by the trace of the normal derivative of the double layer potential on \( \partial \Omega \); that is for smooth functions,
\[
(N \phi)(x) = \frac{\partial}{\partial v} \int_{\partial \Omega} \phi(y) \frac{\partial}{\partial v(y)} \Phi(x, y) \, ds(y), \quad x \in \partial \Omega.
\]
The operator \( G_N : H^{-1/2}(\partial \Omega) \rightarrow L^2(S^2) \) is now the operator which maps the Neumann boundary data \( f \) onto the far field pattern \( v^\infty \) of the corresponding solution of the exterior Neumann boundary value problem.

Adding (3.3) to (3.11) yields
\[
F + F_{x_0} = -[G_N N^* G_N^* + G_{x_0} T_{x_0}^* G_{x_0}^*].
\]
(3.12)
The analogous results of lemma 3.1 and 3.2 are formulated in the following lemma without proof.
Lemma 3.6.

(a) The operator $G_{\lambda_0} : H^{-1/2}(\partial B) \to L^2(S^2)$ can be written in the form

$$G_{\lambda_0} = G_N R_2$$

where $R_2 : H^{-1/2}(\partial B) \to H^{-1/2}(\partial \Omega)$ is given by $R_2 f = \left( \frac{\omega}{\Gamma} + i \lambda v \right) |_{\partial \Omega}$ and $v$ solves the exterior impedance boundary value problem (2.1)–(2.2). Furthermore, $R_2$ is compact.

(b) Set $A_2 := N + R_2 T_{\lambda_0}^* R_2^* : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)$. Then $\mathfrak{Re} (A_2 \phi, \phi) > 0$ for all $\phi \in H^{1/2}(\partial \Omega)$, $\phi \neq 0$.

Substituting $G_{\lambda_0} = G_N R_2$ into (3.12) yields the factorization

$$F + F_{\lambda_0} = -G_N [N^* + R_2 T_{\lambda_0}^* R_2^*] G_N^* = -G_N A_2^* G_N^*.$$  \hfill (3.13)

We check the assumptions of theorem 1.1 again. Assumptions (3) and (4) hold by the previous lemma. Furthermore, $-N$ is a compact perturbation of a coercive operator as shown in [8], theorem 1.26. Therefore, theorem 1.1 is applicable and yields that the ranges of $G_N$ and $F_{\lambda_0}^{1/2}$ coincide where $F_2$ is defined by (3.10) again. Also, lemma 3.3 holds for the solution operator $G_N : H^{-1/2}(\partial \Omega) \to L^2(S^2)$ of the exterior Neumann boundary value problem (simple modification of the proof of theorem 1.12 of [8]). Therefore, theorems 3.4 and 3.5 hold literally.

We finally add two general comments on the factorization methods which apply to all versions.

Remark 1. We comment on the original form where $F_2$ is given by $F_2 = |\mathfrak{Re} F| + \mathfrak{Im} F$. The goal to determine only the shape from the full set of far field patterns seems to be very modest. Note that in $\mathbb{R}^3$ the data $u^{\omega}(i, \theta)$ are parametrized by four parameters while the unknown surface $\partial \Omega$ is parametrized by only three parameters. However, in contrast to all other methods for solving inverse scattering problems nothing has to be known (except the wave number and—in the original version—the fact that it is not an eigenvalue of the corresponding interior eigenvalue problem) about the scattering model. Therefore, the factorization method provides a very general uniqueness result: it cannot happen that the far field patterns corresponding to two different shapes coincide even if the boundary conditions, transmission conditions or interior model equations are different. In order to determine $\partial \Omega$ one has just to look for solvability of the equation $F_{\lambda}^{1/2} g = \phi$; that is, only $F_2$ and nothing else has to be known. We refer again to [8] for more examples and to [10] for the case where the interior model is an elastic equation.

Remark 2. The second remark concerns the ill-posedness of the inverse scattering problem. It is well known that ill-posed problems have to be regularized. We did not do this in our presentation because the obstacle is characterized by the solvability of an equation in $L^2(S^2)$ which is—for classical inverse problems—always assumed a priori for non-perturbed data. It is quite standard to look at regularized solutions of the equation $(F_2^{1/2})^{1/2} g = \phi$, if the data $F_2$ is perturbed by $F_2^{1/2}$ with respect to the operator norm in the infinite dimensional space $L^2(S^2)$. In this case one can apply general convergence results see, e.g., [15]. For the more realistic case of finite dimensional data $u^{\omega}(\tilde{x}_j, \tilde{\theta}_i)$, $j, \ell = 1, \ldots, N$, we refer to [12] for a regularization analysis. Our experiences—even for real data, see figure 1.11 in [8]—showed that we got the best results by no further regularization.
4. Numerical examples

Now we turn to present some numerical examples in two dimensions to illustrate the applicability and effectiveness of the modified factorization method.

There are totally three groups of numerical tests to be considered, and they are respectively referred to as SoftDisk, HardKite and HardDiskEllipse. In all the examples, an impedance boundary condition with $\lambda_0 = 1$ on $\partial B$ was imposed. In our simulations, we used the boundary integral equation method to compute the far field patterns $u_{\infty,i}^e(\theta_j, \theta_i)$ and $u_{\infty,i}^e(\theta_j, \theta_i)$, $j, l = 1, \ldots, 64$, with $\theta_j = 2\pi j/64$, for 64 equidistantly distributed incident directions and 64 observation points. These data are then stored in the matrices $F_\Omega \in \mathbb{C}^{64 \times 64}$ and $F_B \in \mathbb{C}^{64 \times 64}$. In the numerical treatment of the integral equations, 128 quadrature points on $\partial \Omega$ were used to generate the measured far field data $F_\Omega$, while 148 quadrature points on $\partial B$ were used to generate the data $F_B$. We further perturb $F_\Omega$ by random noise using

$$F_\Omega^\delta = F_\Omega + \delta \|F_\Omega\| \frac{R_1 + R_2i}{\|R_1 + R_2i\|},$$

where $R_1$ and $R_2$ are two $64$-by-$64$ matrixes containing pseudo-random values drawn from a normal distribution with mean zero and standard deviation one. The value of $\delta$ used in our code is $\delta := \|F_\Omega^\delta - F_\Omega\|/\|F_\Omega\|$ and so presents the relative error.

In the simulations, we used a grid $G$ of 151 $\times$ 151 equally spaced sampling points on the rectangle $[-4, 4] \times [-4, 4]$ for the first two examples and on the rectangle $[-6, 6] \times [-6, 6]$ for the third example. Let $\{(\sigma_n, \psi_n) : n = 1, \ldots, 64\}$ represent the eigensystem of the matrix $F_\Omega$ given by

$$F_\sigma = |\Re(F_\Omega + F_B)| + |\Im(F_\Omega + F_B)|.$$

Therefore, for each point $z \in G$ we define the indicator function

$$W(z) := \sum_{n=1}^{64} \frac{|\phi_n^e \psi_n^e|^2}{|\sigma_n|},$$

where $\phi_n = (e^{-ik_0z}, e^{-ik_0z}, \ldots, e^{-ik_0z})^T \in \mathbb{C}^{64}$. The values of $W(z)$ should be much larger for the points belonging to $\Omega$ than for those lying in the exterior $\Omega_0$.

**Example SoftDisk**

As shown in figure 1, $\Omega$ is given as a unit disk centered at the origin. We impose the Dirichlet boundary condition on $\partial \Omega$. The wave number $k = 2.404 \times 105 695 772$ is chosen such that $k^2$ is an interior Dirichlet eigenvalue of $-\Delta$ in $\Omega$. In figure 2 we show reconstructions of the original factorization method on the above row and the modified factorization method on the below row. Clearly, the original factorization method indeed fails when the interior eigenvalue problem occurs in the noise-free case. However, the original factorization method with a small amount of noise (even only 0.1%) gives us some *a priori* information on the shape and location of the underlying scatterer $\Omega$. We guess that it may be due to the noisy far field data $u_{\infty,i}^e$ being the far field data for a perturbed domain $\Omega^\delta$ in which $k^2$ is NOT an interior Dirichlet eigenvalue of $-\Delta$.

**Example HardKite**

The example aims to show that the modified factorization method is stable at interior eigenvalues. Again, we will also compare it with the original factorization method. $\Omega$ is given as a kite-shaped domain with boundary $\partial \Omega$ illustrated in figure 3 and described by the
Figure 1. The original domain: unit circle.

Figure 2. Reconstructions using the original factorization method without noise (above left), with 0.1% noise (above middle) and 1% noise (above right) and using the modified factorization method with a smaller artificial disk $B$ (from below left to right, the radiuses of $B$ are 0.01, 0.1 and 0.5, respectively).

parametric representation

$$x(t) = (\cos(t) + 0.65 \cos(2t) - 0.65, 1.5 \sin(t)), \quad 0 \leq t \leq 2\pi.$$  

We impose the Neumann boundary condition on $\partial \Omega$. $k = 1.1136$ is an appropriate value such that $k^2$ is an interior Neumann eigenvalue of $-\Delta$ in the kite shaped domain $\Omega$. In figure 4 we show reconstructions of the original factorization method on the above row and the modified factorization method on the below row.
Figure 3. The original domain: ‘kite’.

Figure 4. Reconstructions using the original factorization method (above row) and using the modified factorization method (below row). Here, the artificial obstacle $B$ is chosen as an ellipse with axes 0.8, 0.6 and center $(0, 0)$.

Figure 5. The original domain: unit disk and ellipse.
Figure 6. Reconstructions using the original factorization method without noise (a), 1% noise (b) and using the modified factorization method (c) with an artificial obstacle $B$. Here, the artificial obstacle $B$ is a disk with radius 0.3 and center $(-4, -4)$.

Example HardDiskEllipse

In this example, the underlying scatterer has two disjoint components. We choose $k$ such that $k^2$ is an interior eigenvalue of $-\Delta$ in one part but not in the other part. As shown in figure 5, $\Omega$ is given as the union of two disjoint obstacles $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1$ is an ellipse with axes 0.8, 0.6, and center $(3, 3)$, while $\Omega_2$ is an unit disk centered at $(-4, -4)$. Again, we impose the Neumann boundary condition on $\partial \Omega$. We take the wave number $k = 3.831 705 970 207 512$ such that $k^2$ is an interior Neumann eigenvalue of $-\Delta$ in $\Omega_2$ and also in $\Omega$. Note that the far field operator $F$ is injective in this case. In figure 6 we show reconstructions of the original factorization method and the modified factorization method. As shown in figure 6(a), the disk part was not correctly recovered by the original factorization method without noise. By comparing the figures 6(b) and (c), we found that the modified factorization method provides a better reconstruction than the original factorization method.

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