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Problem Sheet 5

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Problem 1: Formulate the following problem in weak form and prove that there exists a unique weak solution.

$$-\Delta u(x) + (3 + 2\sin(x))u(x) = f(x) \text{ for } x \in \Omega \subset \mathbb{R}^d \text{ open,}$$
$$u(x) = g(x) \text{ for } x \in \partial\Omega$$

with $f \in L^2(\Omega), g \in L^2(\partial\Omega) \cap TH^1(\Omega)$, where $T : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator.

Problem 2: Show that for $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $\Omega \subset \mathbb{R}^d$ open and bounded the following two statements are equivalent:

$$-\nabla \cdot (A(x)\nabla u(x)) + b(x)u(x) = f(x) \text{ for } x \in \Omega$$
$$\nu(x) \cdot (A(x)\nabla u(x)) = 0 \text{ for } x \in \partial\Omega$$

and

$$a(u, \phi) = l(\phi) \text{ for all } \phi \in C^\infty(\Omega),$$

where $a(u, \phi) = \int_\Omega \nabla\phi(x) \cdot (A(x)\nabla u(x)) + b(x)u(x)\phi(x)dx$ and $l(\phi) = \int_\Omega f(x)\phi(x)dx$.

Problem 3: For $\Omega \subset \mathbb{R}^d$ open and bounded derive the weak formulation of

$$-\nabla \cdot (A(x)\nabla u(x)) + b(x)u(x) = f(x) \text{ for } x \in \Omega$$
$$u(x) = g(x) \text{ for } x \in \Gamma_D$$
$$\nu(x) \cdot (A(x)\nabla u(x)) = 0 \text{ for } x \in \Gamma_N,$$

where $\Gamma_D \cup \Gamma_N = \partial\Omega, \Gamma_D \cap \Gamma_N = \emptyset$.

Hint: The solution space should be the closure of $H^1(\Omega) \cap \{u \in C^\infty(\Omega) : u|_{\Gamma_D} = 0\}$ *with respect to the* H^1*-norm.*